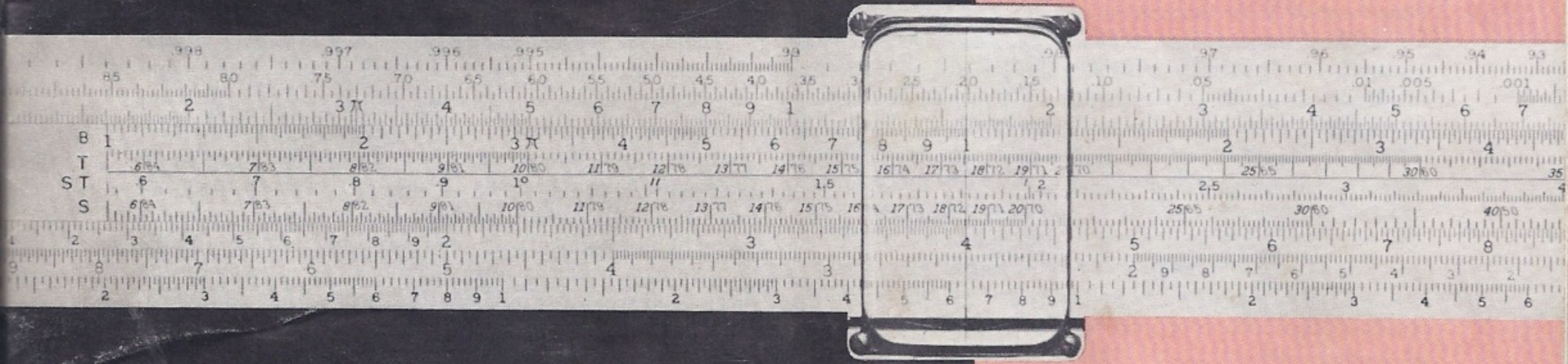
Electrical Manufacturing

Slide Rule Mathematics

A practical guide to the understanding of the slide rule and its application to engineering problems



IRA RITOW, *Guided Missile Division*
REPUBLIC AVIATION CORPORATION
Farmingdale, N. Y.

THE SLIDE RULE is *the* basic tool of the engineering profession; however, too few engineers use this versatile instrument to its fullest advantage. Here are a logical development of the slide rule's fundamentals and complete, concise instructions for its use.

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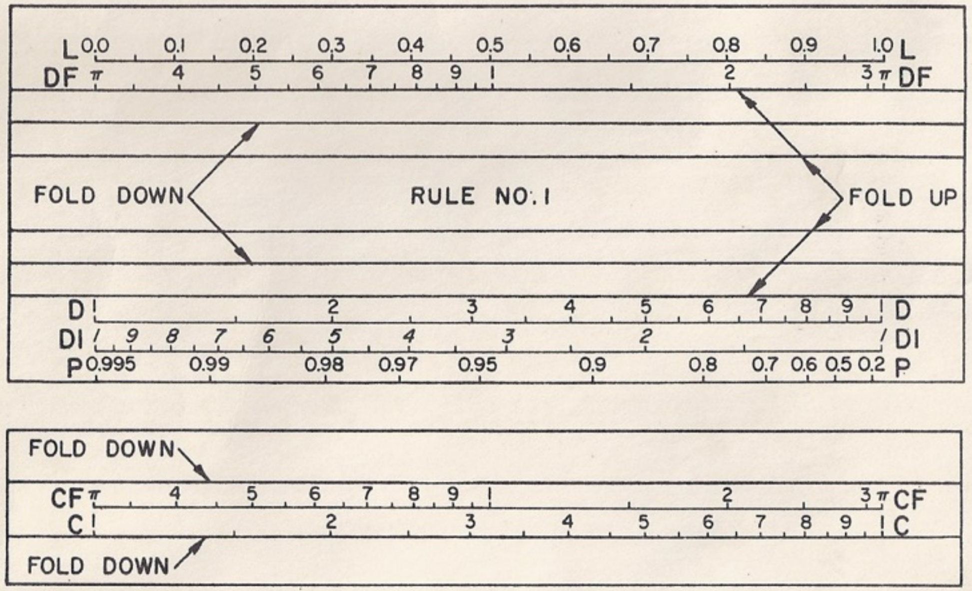

Fig. 1—Practice Slide Rule No. 1.

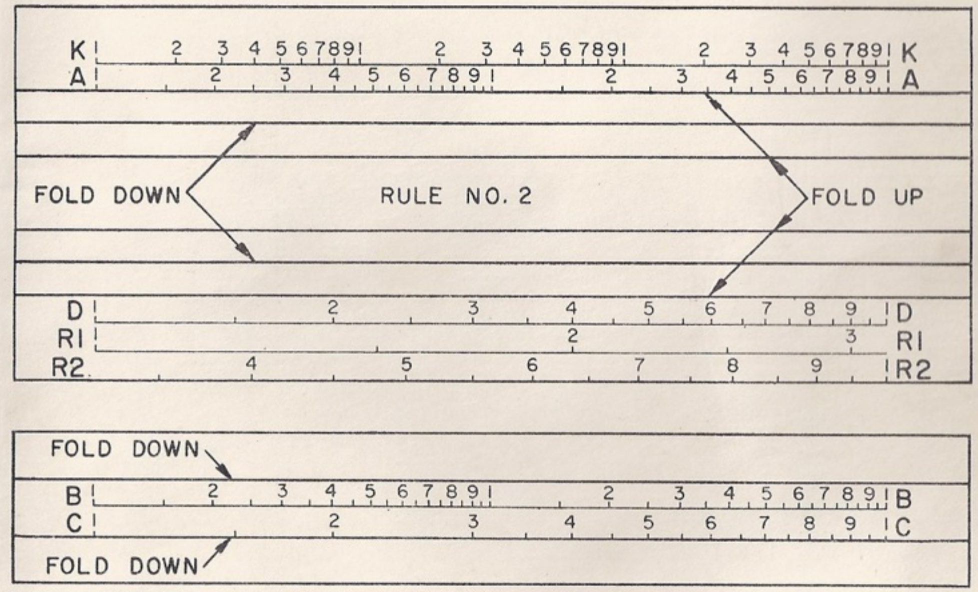

Fig. 2—Practice Slide Rule No. 2.

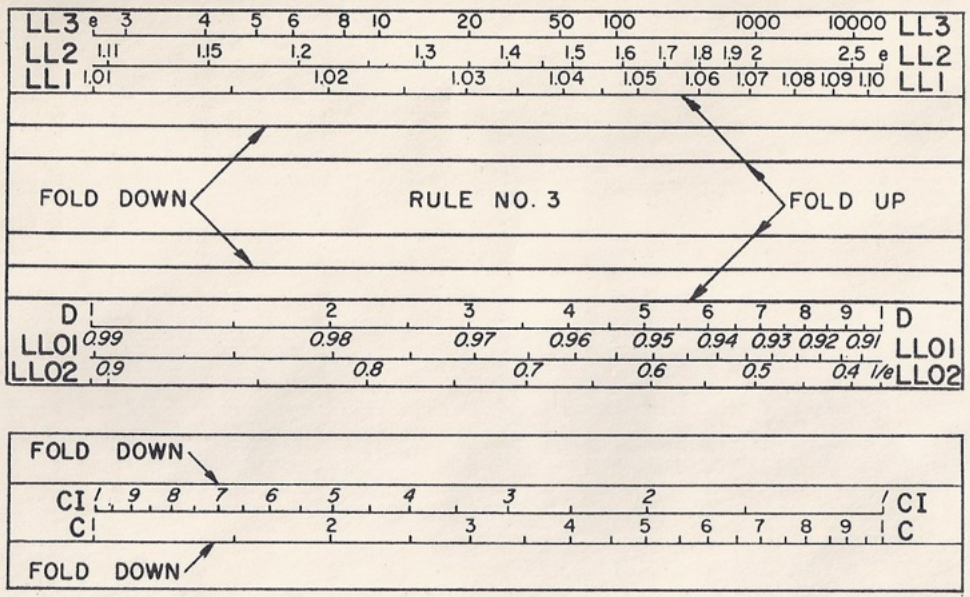

Fig. 3—Practice Slide Rule No. 3.

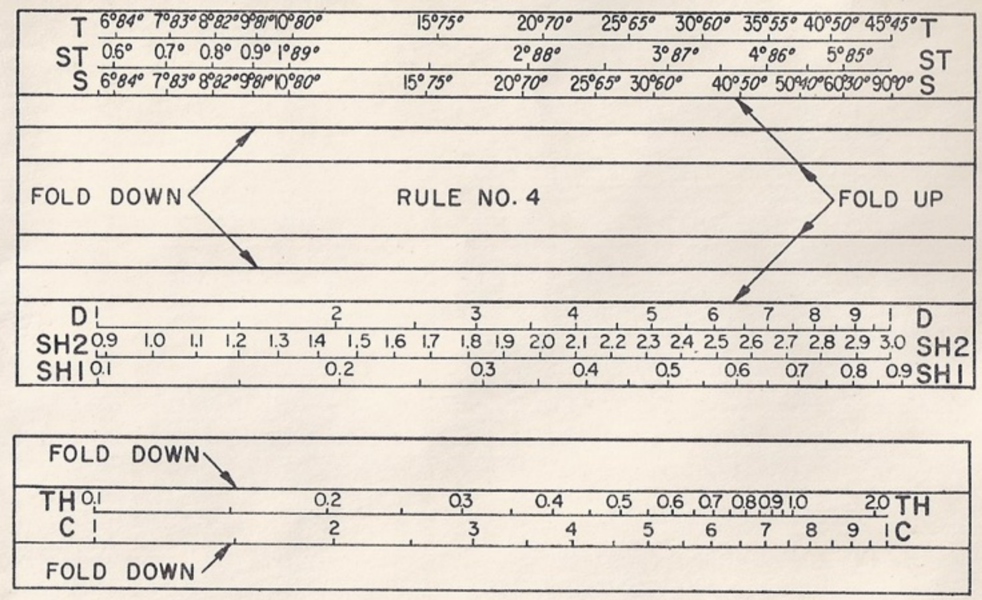

Fig. 4—Practice Slide Rule No. 4.

The Basic Slide Rule

WHY USE THE SLIDE RULE? The slide rule is a computing device. As a computing device, it has such dissimilar relatives as: a pencil and paper, an abacus, a desk calculator and a general purpose analog computer. The user of a computing device must consider accuracy, speed and cost in selecting his weapon. The slide rule is fast, versatile and cheap. Its accuracy of $\pm \frac{1}{2}$ per cent or better on most problems is more than adequate for almost all engineering calculations. It is portable and relatively foolproof. Even accountants, the last bulwark of the paper and pencil advocates, are among the slide rule fans today. For non-repetitive calculations, it can hardly be surpassed. The modern slide rule is the result of the cascaded efforts of many men, from John Napier to Herman Ritow (the author's father), whose combined ingenuity over a period of four centuries has resulted in the modern slide rule.

Adding Logarithms to Multiply. The basic slide rule is a scheme that solves multiplication problems by adding distances on a scale. This scheme will now be explained.

Suppose it is desired to multiply 12×4. (12 and 4 are called "factors".) The usual method of multiplication involves successive addition, such as:

$$\left.\begin{array}{r}12\\12\\12\\12\end{array}\right\}\text{4 times}$$
$$\overline{48}$$

The word "times" for multiplication is a clue to the process of adding one factor into a sum a certain "number of times" corresponding to the second factor. Desk calculators operate on the "times" principle.

The slide rule multiplies by a completely different principle. 4×12 can be multiplied as follows:

$$4 \times 12 = 10^{(0.602)} \times 10^{(1.078)} = 10^{(0.602+1.078)} = 10^{(1.68)} = 48$$

In this method each factor is rewritten as 10 raised to some exponential power. These exponents are then added and the resultant power of ten is the desired answer. As a first step toward making a slide rule, a graph of ten raised to various exponential powers is shown in Fig. 5. Using this graph to solve 4×12, the following steps are necessary:

1. Convert 4 and 12 to exponential powers of ten. This is done by finding points *b* and *d* on the graph. Dashed lines *ab* and *cd* represent the exponents of ten that correspond to 4 and 12 respectively.
2. Add the exponents. To add the exponents, it is only necessary to draw line segment *bd'* as shown so that *bd'* has the same length as line segment *cd* and note the sum, line segment *ad'*.
3. Convert the exponential sum into a number. Line segment *ad'* now represents the sum of the exponents of ten that in turn represented the original factors, 4 and 12. Converting distance *ad'* to a number by moving the point *d'* across to intersect the curve at point *e* gives the product desired, 48.

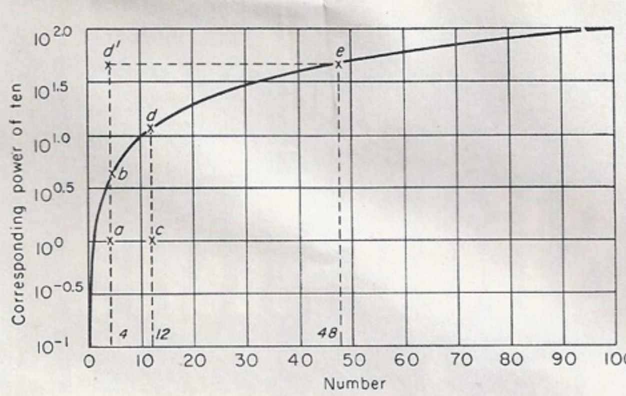

Fig. 5—Conversion chart for obtaining corresponding power of ten for any number.

The above scheme using Fig. 5 has a drawback in that the numbers close to zero are difficult to read. As a second step toward developing a slide rule, Fig. 6 is a redrawn version of Fig. 5 with the horizontal scale distorted so that the graph is a straight line. Again line segment *ab* represents 4, *cd* represents 12 and *cd'* represents the product (4×12). Notice, however, that since the graph is a straight line, segments a_1b_1 and c_1d_1 could have been added to give $a_1d'_1$ which is read directly as 48. Therefore, the horizontal scale of Fig. 6 by itself will permit multiplication by adding distances!

Figure 7 shows the final step in the development of the slide rule. The distorted number scale of Fig. 6 has been

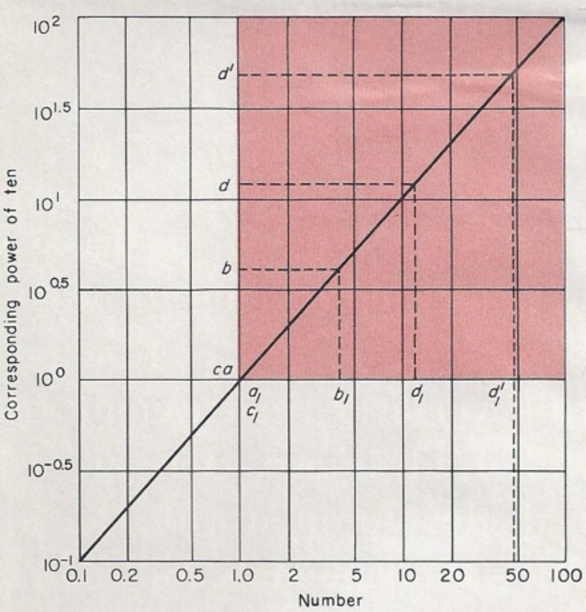

Fig. 6—Conversion chart of Fig. 5 distorted to obtain straight line.

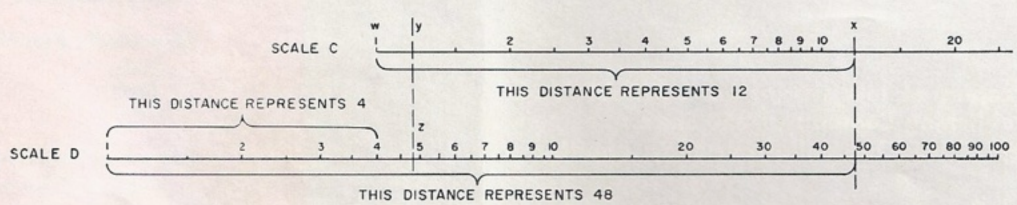

Fig. 7—Elements of slide rule multiplication.

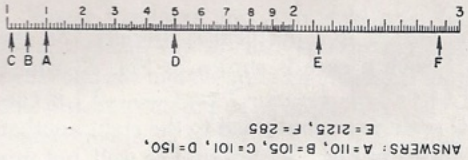

Fig. 8—Scale reading exercise.

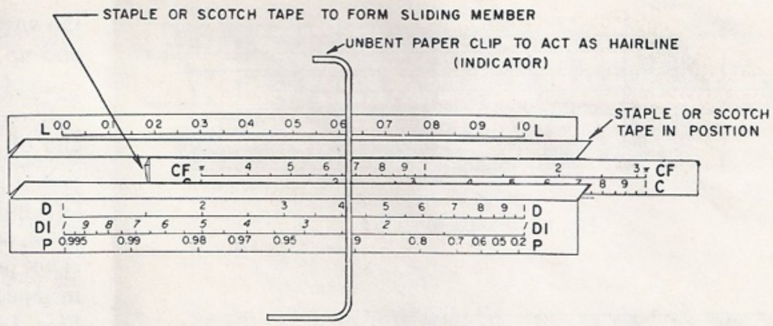

Fig. 9—Construction of slide rule training aid.

drawn twice in Fig. 7, as scale C and scale D. Distances representing 4 and 12 are added as before to give the product, 48. Notice that it would be very convenient to have scale C movable left and right so that any set of distances could be conveniently added. The C scale is movable in a slide rule. Notice also that the scale from 10 to 100 has the same layout as the scale from 1 to 10. As a result, y has the same relative position on the C scale as x and using distance wy instead of wx in the multiplication problem converts the problem to 4×1.2 instead of 4×12. The answer to 4×1.2 appears at z on scale D; $z = 4.8$. In order to obtain maximum accuracy in a calculation, slide rules are made with only one "cycle" (i.e., 1→10) in both C and D scales and decimal places are shifted as needed to make the answer appear on the scale. Consequently, *the slide rule does not give decimal places.** This will be demonstrated in later examples.

In solving problems with the slide rule, the user must insert the decimal point in the answer himself. Another mental burden that he faces is reading the scales on the slide rule. This takes more practice than most people will admit. Because of the uneven scale spacing, the smallest division will be $\frac{1}{5}$, $\frac{1}{2}$ or $\frac{1}{10}$ of a major division depending on the position on the scale. Readings just beyond 1.00 are also troublesome. As a limbering up exercise, try reading the pointer settings in Fig. 8 as rapidly as possible. The correct answers are shown inverted at the bottom of the figure.

The term logarithm has been avoided up to this point to avoid a common form of panic that this term seems to catalyze. For completeness and for use in later sections, the logarithm is introduced here. Logarithms are most easily explained by this equation:

$$a = 10^b$$

If a and b are numbers, then *b is the logarithm of a to the base 10.*

It is apparent from this definition that the multiplication-by-adding scheme developed so far was really a means of multiplying factors by adding their logarithms. Another way of defining a logarithm is to state: The logarithm of a number to some base is the exponent to which the base must be raised to equal the given number. A more basic definition of logarithms in terms of an infinite series exists, but will not be given here.

In adding distances on the slide rule, "1" is used as the starting point or "zero" distance because the logarithm of one is zero. A look back at Fig. 6 shows that ten raised to the zero power and the number one were used as base lines for distance measurement. A hairline or indicator is used with the sliding scales in a slide rule so that points on two different scales may be readily aligned.

Sample Slide Rules. Figure 1 shows a do-it-yourself slide rule, the first of four rules that can be cut out of this article to form training aids that will permit the reader to "work along" with the examples. Cut out the two parts of Fig. 1 along their outside outlines, fold as directed, and staple or scotch tape each part separately. Then lay the slide inside the body as shown in Fig. 9. An unbent paper clip laid loosely on top of the rule can be used as a hairline.

Figs. 2, 3 and 4 show similar rules. The set of four will permit the reader to practice each example as it is described and this practice is highly recommended. Each rule has a code number and all illustrative examples indicate the appropriate rule to use. These rules are designed to simplify and speed learning and do not represent the best choice in scale arrangements.

Multiplication. Fig. 10 shows simple multiplication, $2 \times 4 = 8$ [(···) + (− − −) = (——)], using the C and D scales. The dots and dashes after the equation key the slide rule manipulation to the equation and indicate the distance addition that corresponds to the multiplication. The vertical lines across the slide rule show the positions of the hairline. The dashed hairline is an intermediate setting used to align 2 on the D scale with 1 on the C scale, and the solid hairline shows the final setting for reading the answer which is circled in red. Follow along each example with your practice slide rule.

Fig. 11 shows another equally proper way of setting up the same example since order is immaterial in multiplication.

Although most people manipulate the slide rule to leave

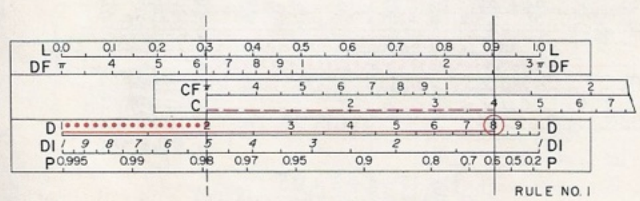

Fig. 10—2 X 4 = 8

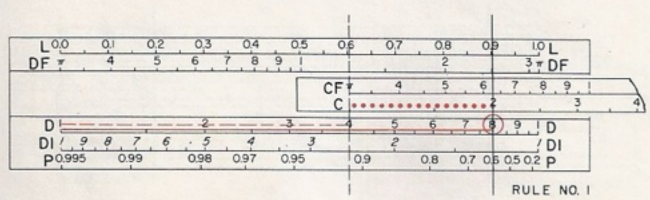

Fig. 11—2 X 4 = 8

*Methods have been devised to keep track of the decimal place during slide rule manipulation, but although ingenious, they are easily remembered wrong and few engineers use them.

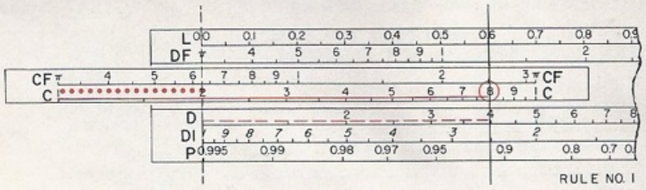

Fig. 12—2 X 4 = 8

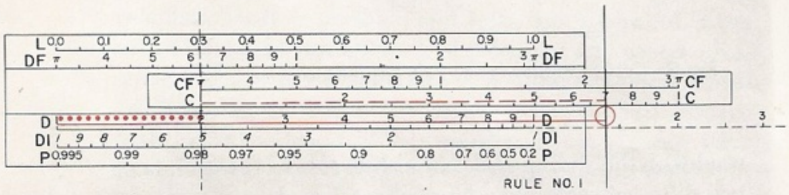

Fig. 13—2 X 7.5 = 15

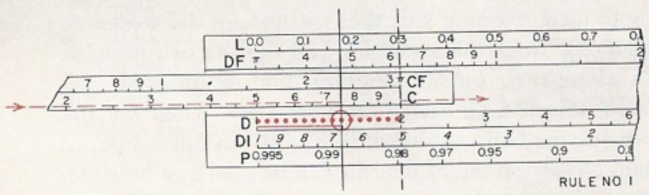

Fig. 14—2 X 7.5 = 15

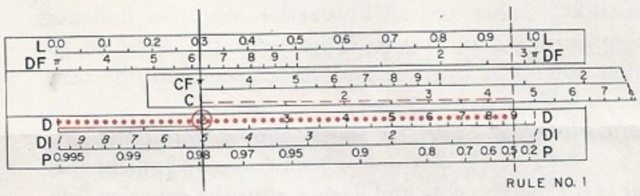

Fig. 15—9 ÷ 4.5 = 2

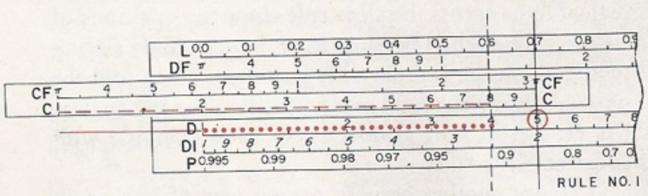

Fig. 16—4 ÷ 8 = 0.5

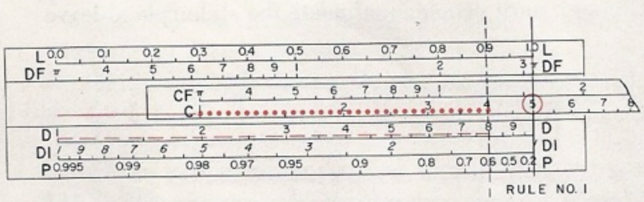

Fig. 17—4 ÷ 8 = 0.5

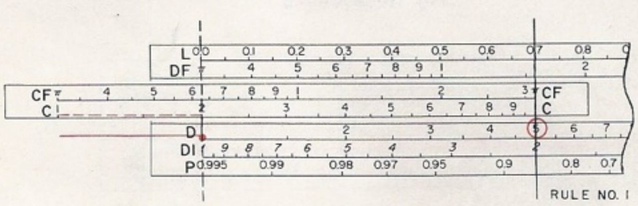

Fig. 18—½ = 0.5

the answer on the D scale, it is perfectly proper to have it end up on the C scale as shown in Fig. 12 for the same example.

Sometimes the answer to a problem is literally "off the end of the rule". Such a case is shown in Fig. 13 where $2 \times 7.5 = 15$ [(· · ·) + (– – –) = (——)] is carried out. If the D scale were continued to the right another cycle, as shown dotted in Fig. 13, all would be well, but this is not possible. The way out is to shift the C scale to the left, instead, as shown in Fig. 14. The reader should compare Figs. 13 and 14 to prove to himself that they are equivalent. In fact, if the C and D scales repeated themselves several times, such as is done in part in Fig. 7, there would be one point in each cycle where the hairline could be located to read out the answer. The shift of the slide by a whole cycle puts the right hand "1" of the C scale in the former place of the left hand "1" of the C scale. This process is called "interchanging indices". "One" on the C or D scale is called an index because it is the starting place for distance measurements. It corresponds to the heavy vertical axis of Fig. 6.

Division and Reciprocals. Division is accomplished on the slide rule by subtracting distances. This can be seen from:

$$10^a/10^b = 10^{(a-b)}$$

In words, the quotient is ten raised to the difference between the exponents when divisor and dividend are both expressed as a number raised to various powers.

Division is illustrated in Fig. 15 for the case, $9 \div 4.5 = 2$ [(· · ·) – (– – –) = (——)]. Notice the dotted intermediate hairline setting. A second example $4 \div 8 = 0.5$ [(· · ·) – (– – –) = (——)] is shown in Fig. 16. The answer appears off in space under the left hand C scale index. Again, because of the cyclic nature of the C and D scales, the same numerical value appears under the right hand index of the C scale and it is read there.

As in multiplication, the roles of the C and D scales can be reversed. Fig. 17 shows the above problem solved with scale roles reversed. In this case, the right hand D scale index substitutes for the left hand D index.

Reciprocals are easily formed by division. For example, ½ can be rewritten as $1 \div 2 = 0.5$ [(· · ·) – (– – –) = (——)] as shown in Fig. 18. A more elegant means of finding reciprocals will be covered later.

Repeated and Mixed Processes. Repeated products are readily calculated by using the hairline to store intermediate products. For example $4 \times 2 \times 7.5 \times 25 = 1500$ is solved in these stages:

$4 \times 2 = 8$ [(· · ·) + (– – –) = (——)]

Fig. 19A

$8 \times 7.5 = 60$ [(· · ·) + (– – –) = (——)]

Fig. 19B

$60 \times 25 = 1500$ [(· · ·) + (– – –) = (——)]

Fig. 19C

Notice that, as the slide is repositioned to introduce each factor in turn, the hairline remains fixed to keep track of the intermediate product.

Repeated quotients are calculated by a similar scheme. For example, $12 \div 8 \div 75 = 0.02$ is solved in these stages:

$12 \div 8 = 1.5$ [(· · ·) – (– – –) = (——)]

Fig. 20A

$1.5 \div 75 = 0.02$ [(· · ·) – (– – –) = (——)]

Fig. 20B

In this case the hairline keeps track of intermediate quotients while the slide is moved.

Mixed products and quotients are also calculated by using the hairline to store intermediate products or quotients and are most readily calculated if quotients and products are alternated. For example: $9 \times 12 \times 5 \div 30 \div 6$ is rearranged to $12 \div 30 \times 9 \div 6 \times 5$ and is solved in the following stages:

$12 \div 30 \times 9 = 3.6$ [(•••) − (– – –) + (–•–•) = (——)] Fig. 21A

$3.6 \div 6 \times 5 = 3.0$ [(•••) − (– – –) + (–•–•) = (——)] Fig. 21B

Variations on the Basic C-D Operation

Displaced Scales. Displaced scales* are variations of the basic C and D scales. Fig. 22 shows two ordinary (C or D) slide rule scales displaced from each other by a distance that corresponds to π. Notice that since the lower scale index is π distance to the right of the upper scale index, $2 \times \pi$ may be read on the upper scale as 6.28 directly above 2 on the lower scale. Similarly 3π appears on the upper scale directly above 3 on the lower scale. As a result, two scales with indices displaced by π provide a rapid means of multiplying by π. Another use of these scales is to read on the lower scale the value of any number on the upper scale divided by π, such as $25/\pi$ as shown in Fig. 22.

A displaced scale arrangement is shown in Fig. 23. The DF scale is the same as the D scale but has been shifted to the left by π units. The solid hairline shows how to obtain $2\pi = 6.28$ on the DF scale directly above 2 on the D scale. The dotted hairline shows how to obtain $2/\pi = 0.636$ on the D scale directly below 2 on the DF scale. The digits involved in each of these cases are very similar and it is necessary to be careful to remember that the operation is "upward" to multiply by π and "downward" to divide by π. A better method (than memorizing) is to notice that the ends of the rule (see red arrows in Fig. 23) show at a glance that the DF scale reads $\pi \times$ the D scale setting and that the D scale reads $1/\pi$ of the DF scale setting.

An example of the DF scale use is shown in Fig. 24 where $(0.4/\pi) \times 7.5 = 0.954$ [(•••) + (– – –) = (——)] is solved. Notice that $0.4/\pi$ appears on the D scale by putting the hairline over 4 on the DF scale.

A similar problem is $7.5 \times 0.4 \times \pi = 9.44$ [(•••) + (– – –) + (–•–•) = (——)] as shown in Fig. 25. Notice that indices were interchanged on the C scale and that the last distance addition (multiplication by π) was performed by transfer to the DF scale from the D scale.

The existence of a CF scale that is equivalent to the C scale shifted left by a factor of π means that both C and D have first cousins, π removed, so to speak. This feature permits avoiding index interchanges in many problems. Fig. 26 shows $2 \times 7.5 = 15$ [(•••) + (– – –) = (——)] using the CF and DF scales. If 2 on the D scale is combined with 7.5 on the C scale, the answer should appear on the D scale opposite 7.5 on the C scale. This answer is literally off the end of the rule so that using the fact that *at all times CF vs DF is the same as C vs D*, the answer is read on the DF scale opposite 7.5 on the CF scale. Compare this method with Figs. 13 and 14 where the

*Also called "folded scales".

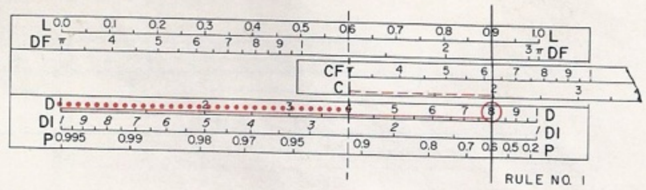

Fig. 19A—4 X 2 = 8

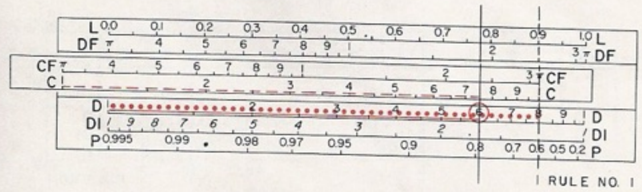

Fig. 19B—8 X 7.5 = 60

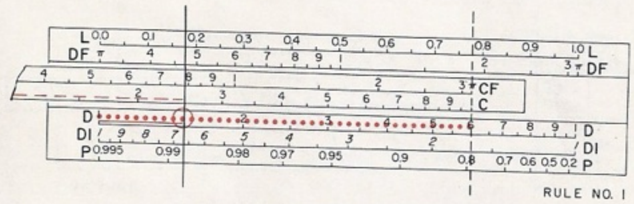

Fig. 19C—60 X 25 = 1500

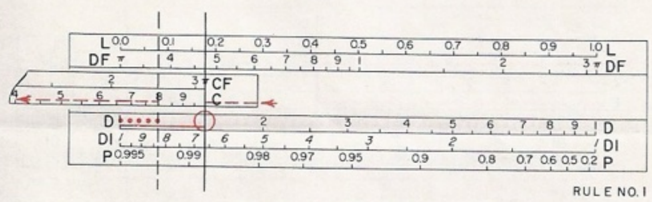

Fig. 20A—12 ÷ 8 = 1.5

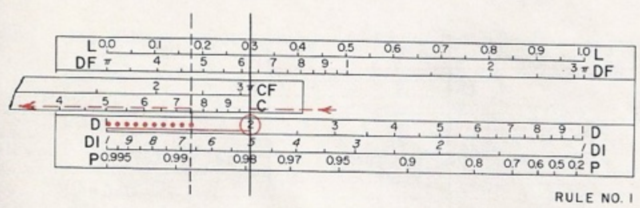

Fig. 20B—1.5 ÷ 75 = 0.02

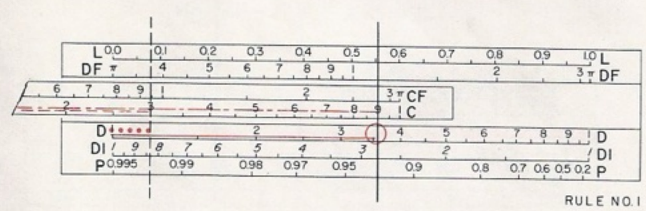

Fig. 21A—12 ÷ 30 X 9 = 3.6

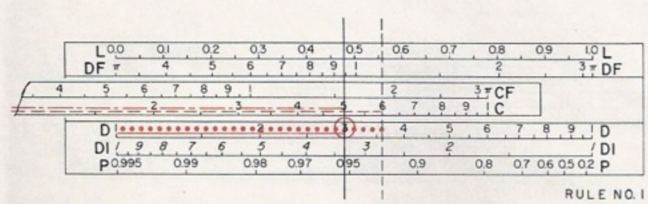

Fig. 21B—3.6 ÷ 6 X 5 = 3.0

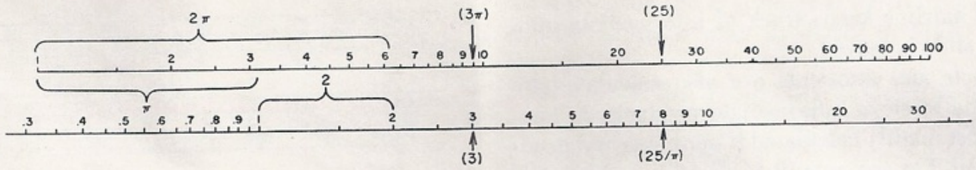

Fig. 22—Displaced scales.

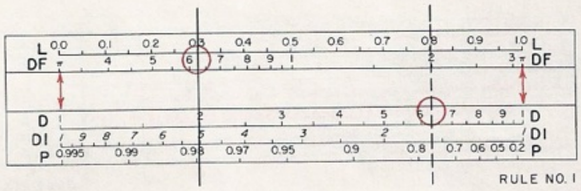

Fig. 23—Displaced scales on slide rule.

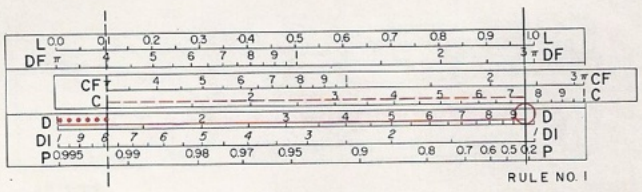

Fig. 24—$(0.4/\pi) \times 7.5 = 0.954$

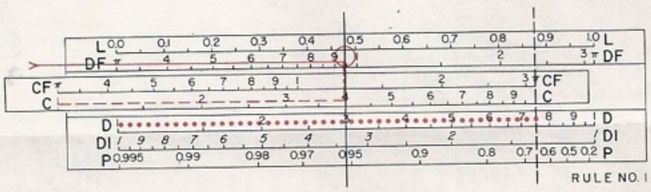

Fig. 25—$7.5 \times 0.4 \times \pi = 9.44$

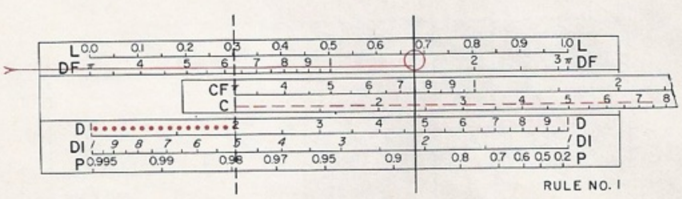

Fig. 26—$2 \times 7.5 = 15$

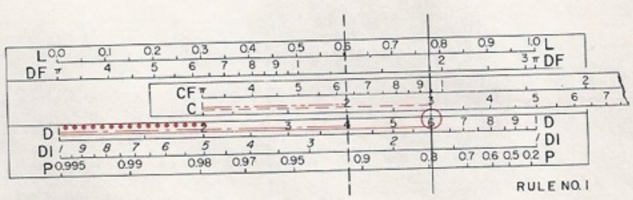

Fig. 27—$x \div 3 = 4 \div 2 = 2$

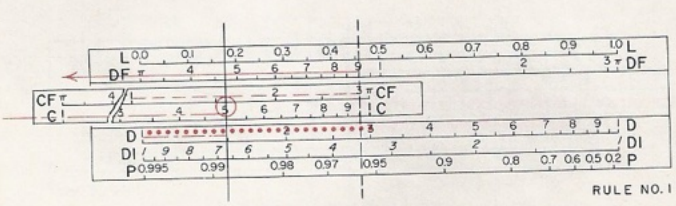

Fig. 28A—$9{:}3 = 1.5{:}x = 3$

method of interchanging indices was used to solve the same problem.

Proportions. Proportions are readily set up on the slide rule. Referring to Fig. 22, it is apparent that, if two similar scales are displaced, then readings on the two scales that correspond in position bear a constant ratio to each other. In Fig. 22, all the upper scale numbers are π times their correspondents. A proportion problem such as, find x if $x{:}3 = 4{:}2$ can be readily solved on the slide rule as shown in Fig. 27. This problem can be thought of as the set-up for $x \div 3 = 4 \div 2 = (2 \div 1) = 2$ [(——)−(– – –) = (–·–·)−(–··–··)=(···)]. First 4 on the D scale is set opposite 2 on the C scale; then x is whatever appears on the D scale opposite 3; in this case, 6.

Since the CF scale is shifted from the DF scale the same amount that the C scale is shifted from the D scale, part of the proportion may appear on the CF and DF scales. Figs. 28A and 28B show two set-ups for $9{:}3 = 1.5{:}x = 3$ [(——) − (– – –) = (–·–·) − (–··–··) = (···)]

Fig. 28C shows another way of looking at this problem, or any problem in proportions. That is $A{:}B = C{:}x$.

Inverted Scales. Inverted scales are used to obtain reciprocals of numbers. The DI scale is an inverted D scale. Fig. 29 illustrates how the DI scale is derived with the problem $1 \div 5 = 0.2$ [(···) − (– – –) = (——)]. Notice that 5 on the DI scale is directly below 2, its reciprocal, on the D scale. By repeating $1 \div x = ?$ for all values of x, the entire DI scale would be generated. To find the reciprocal of any number on the D scale, read the DI scale below; or to find the reciprocal of any number on the DI scale, read the D scale above. For example, Fig. 30 solves $(1/6) \times 3 = 0.5$ [(···) + (– – –) = (——)].

Because the DI scale is the same scale as the D scale but reads right to left, the distance from the left index to a number on the D scale is the same as the distance from the right index to the same number on the DI scale as shown in Fig. 31, for the number 2.5.* Using this principle, Fig. 32 shows $2 \times 4 = 8$ [(···) + (– – –) = (——)] which should be compared with the same problem solved without using the DI scale in Fig. 12.

A slide rule with a CI scale which is an inverted C scale permits successive multiplication or division with few or no slide movements while leaving the answer on the D scale. For example, $4 \times 2 \times 7.5 = 60$ [(···) + (– – –) + (–·–·) = (——)] is solved with one slide setting in Fig. 33 although the same problem required two slide settings in Figs. 19A and 19B.

Squares and Square Roots. If a distance on the D scale called z is halved, the half distance, call it y, represents a number such that $y \times y = z$. This is illustrated in Fig. 34 where $z = 9$ and $y = 3$. Since $y \times y = y^2 = z$, then $y = \sqrt{z}$. Notice that the solid hairline falls on 9 on

*The numbers of the DI scale are italicized to act as a reminder that they read from right to left. On some slide rules the reverse reading scales are colored.

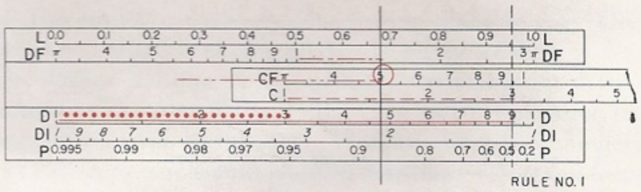

Fig. 28B—9:3 = 1.5: $x = 3$

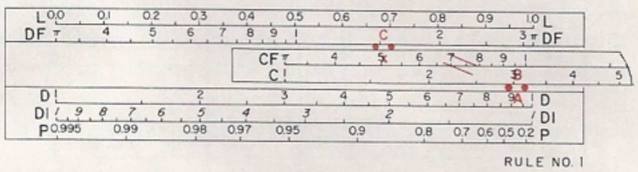

Fig. 28C—A:B = C:x

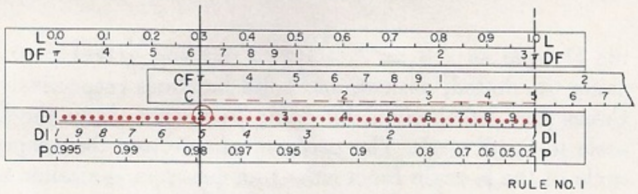

Fig. 29—1 ÷ 5 = 0.2

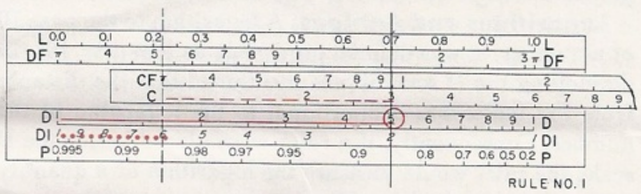

Fig. 30—(1/6) X 3 = 0.5

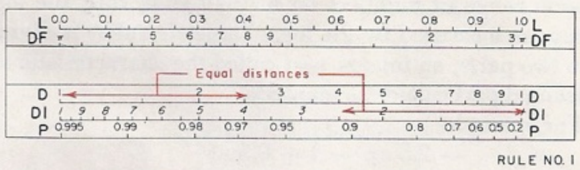

Fig. 31—Equal distances on inverse scales.

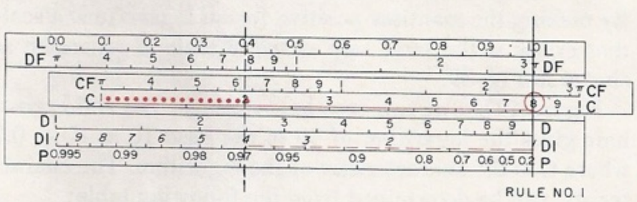

Fig. 32—2 X 4 = 8

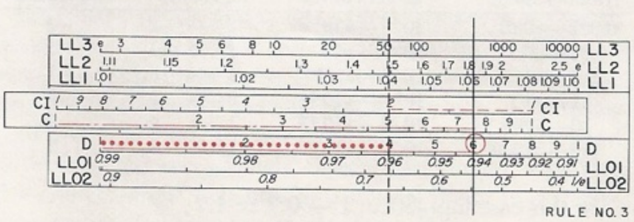

Fig. 33—4 X 2 X 7.5 = 60

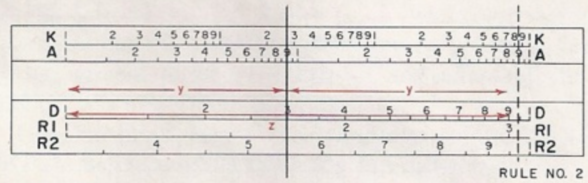

Fig. 34—y X y = z

To find the proper scale for $\sqrt[2]{y}$ or $\sqrt[3]{y}$

I. Rewrite y as $C10^N$ where C is between 1.00 and 9.9999.....

II. For squares and square roots,

If N is	use	or
even	R_1 scale	left A scale
odd	R_2 scale	right A scale

III. For cubes and cube roots,

If N is ±	and divisable by 3 with	left over, use
	0	left K scale
+	1	center K scale
+	2	right K scale
−	1	right K scale
−	2	center K scale

Fig. 35—Conversion chart for root scales.

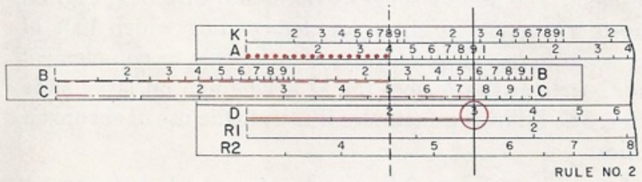

Fig. 36A—$\sqrt{4} \div \sqrt{25}$ X 7.5 = 3

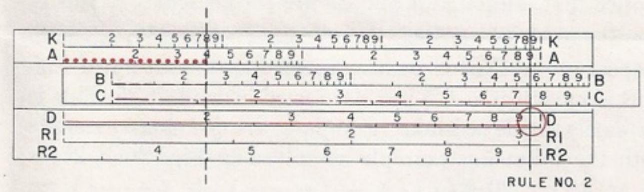

Fig. 36B—Incorrect setup for $\sqrt{4} \div \sqrt{25}$ X 7.5 = 3

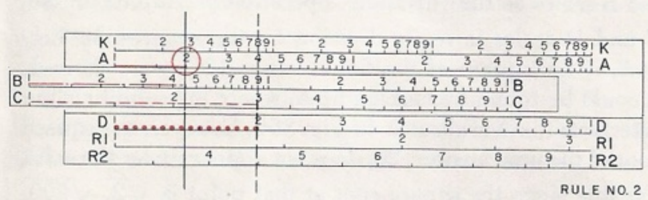

Fig. 36C—$2^2 \div 3^2$ X 45 = 20

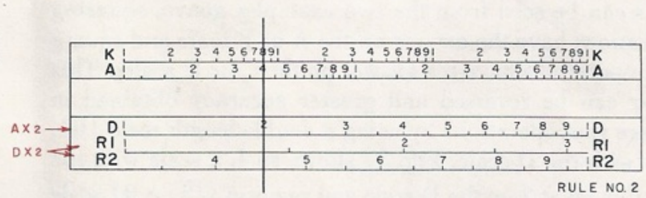

Fig. 37—R1 and R2 are double length square root scales.

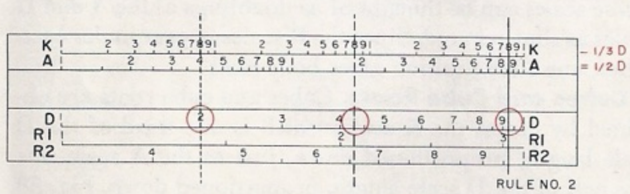

Fig. 38—Cube roots of 8, 80 and 800.

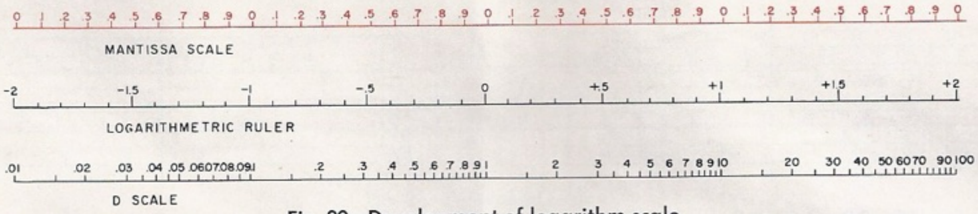

Fig. 39—Development of logarithm scale.

the A scale. The A scale is a shrunken (by ½) D scale and so it indicates y^2 above the corresponding value y on the D scale. Consequently, the square of any number on the D scale appears on the A scale and the square root of any number on the A scale appears on the D scale. The dashed hairline of Fig. 34 shows that 9.48 on the D scale is also a square root of 9!! This is not true for 9 but is true for 90. The square root of any number depends on the position of the decimal point. The user has two choices. Either regard the left half of the A scale as being for the ranges $1-10$, $100-1000$, $0.01-0.1$, etc. and the right half for the ranges $0.1-1$, $10-100$, etc. or just mentally calculate the approximate root to decide which half of the scale to use for any problem. The author suggests the latter, however, Fig. 35 shows a conversion chart for determining which half of the scale to use. The addition of a B scale to complement the C scale as the A does the D is common on large slide rules. The following examples illustrate the use of shrunken scales:

$\sqrt{4} \div \sqrt{25} \times 7.5 = 3$ [(···) – (– – –) + (–·–·) = (——)] in Fig. 36A

Notice that, although 4 and 25 are set on scales A and B, the distances so marked out represent $\sqrt{4}$ and $\sqrt{25}$ with respect to the C or D scales. Notice that the set-up of Fig. 36B is incorrect as 2.5 has been used instead of 25. This is an easily made mistake. Compare the previous example with the following example as solved in Fig. 36C: $2^2 \div 3^2 \times 45 = 20$ [(···) – (– – –) + (–·–·) = (——)]. In this case distances are with respect to the A and B scales so the "division" operation performed on the C and D scales is really division by the squares. In Fig. 36A, if the square of the final answer (3^2) were desired, it could be found by looking up to where the solid hairline intersects the A scale at 9. In Fig. 36C, however, the square root of the final answer, 20, does not appear under the solid hairline since the square root at that point is $\sqrt{2}$, $\sqrt{200}$, $\sqrt{0.02}$ etc.

As can be seen from the two examples above, squaring operations have the answer on the A or B scale and square root operations have the answer on the C or D scales. This order can be reversed and greater accuracy obtained in square root operations by using a double length scale (R1, R2) with the D scale. Fig. 37 shows such a scale with the hairline set at 2 on the D scale and reading $\sqrt{2}$ on R1 scale ("Root-1" scale) and $\sqrt{20}$ on the R2 ("Root-2" scale). These scales can be thought of as doublings of the A and D scales as shown in red. Since the R scales appear under each other, one slide position gives both roots.

Cubes and Cube Roots. Cubes and cube roots are obtained by use of the K scale, which is one third of the D scale length, proportioned down, just as the A scale was one half of the D scale length, proportioned down. Fig. 38 shows the three cube roots of the digit 8. They appear on

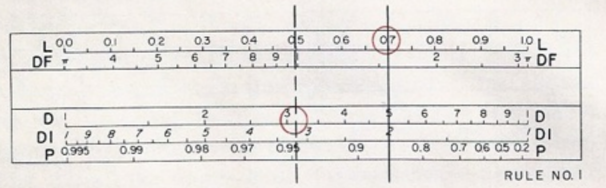

Fig. 40—$\text{Log}_{10} 50 = 1 + 0.7 = 1.7$
$\text{Antilog}_{10} 0.5 = 0.316$

the D scale as $\sqrt[3]{8} = 2$, $\sqrt[3]{80} = 4.3$ and $\sqrt[3]{800} = 9.3$ using the dotted, dashed, and solid hairlines respectively. Cubes may be obtained by going in reverse from the D scale to the K scale. The problem of selecting the proper cycle of the K scale for a cube root problem is similar to the problem of selecting parts of the A scale when square rooting. Again, an approximate mental calculation is suggested although the chart of Fig. 35 can be used.

Logarithms and Antilogs. A logarithm to the base 10 of a number, a, was defined previously as b in $a = 10^b$. In developing the D scale it was mentioned that the distance from the index was proportional to the logarithm of the number. Consequently, if a ruler were placed beside the D scale, the ruler would measure the logarithm or a quantity proportional to the logarithm.

Redrawing the bottom part of Fig. 7 with such a ruler alongside gives the scale combination of Fig. 39. To cover a wide range of numbers on a small slide rule, the logarithm, as measured by the logarithmetric ruler, is divided into two parts, an integer part called the characteristic and a decimal part called the mantissa.

For example:

$$-2.5 = -3 + 0.5$$
$$+2.5 = +2 + 0.5$$
$$+0.2 = +0 + 0.2$$
$$\text{Logarithm} = \text{Characteristic} + \text{Mantissa}$$

By making the mantissa positive for all logarithms, a scale that cycles at the same rate as the D scale is generated as shown in Fig. 39.

Figure 40 shows two crosshair settings. The solid crosshair gives the logarithm of 50 to the base 10 as C + 0.7 where C is the characteristic of the logarithm. The characteristic can be determined from the following table:

Number range	Number range written in powers of ten	Range of logarithm	Characteristic part of logarithm
0.001⟶0.01	$10^{-3} \longrightarrow 10^{-2}$	$-3 \longrightarrow -2$	−3
0.01⟶0.1	$10^{-2} \longrightarrow 10^{-1}$	$-2 \longrightarrow -1$	−2
0.1⟶1	$10^{-1} \longrightarrow 10^{0}$	$-1 \longrightarrow 0$	−1
1⟶10	$10^{0} \longrightarrow 10^{+1}$	$0 \longrightarrow 1$	0
10⟶100	$10^{+1} \longrightarrow 10^{+2}$	$1 \longrightarrow 2$	1
100⟶1000	$10^{+2} \longrightarrow 10^{+3}$	$2 \longrightarrow 3$	2

In this case $\log_{10} 50 = 1 + 0.7 = 1.7$
Similarly $\log_{10} 0.5 = -1 + 0.7 = -0.3$

The antilog can also be found using the slide rule. The antilog_{10} of 0.5 is shown in Fig. 40 on the D scale under the dashed hairline as 316 with the decimal point determined by the table above, in this case 0.316. Ten raised to a power is given by the same procedure, for 10^b is another way of expressing $\text{antilog}_{10}b$. Logarithms to bases other than 10 will be discussed later.

Pythagorean Scale. The log scale is only one of an infinite variety of scales that can be composed to go with the D scale. Another example is the P (for Pythagorean) scale. This scale is used to find the length of the third side of a right triangle if the hypotenuse and opposite side are known.

Fig. 41 shows a right triangle. From geometry,

$$h^2 = a^2 + b^2$$

$$\text{or} \quad \frac{a^2}{h^2} = 1 - \frac{b^2}{h^2}$$

$$\text{or} \quad \frac{a}{h} = \sqrt{1 - \left(\frac{b}{h}\right)^2}$$

where a/h and b/h are the lengths of the sides expressed as fractions of the hypotenuse. If the D scale is thought to go from .1 to 1.0 as indicated in Fig. 42, then the D scale setting is b/h and the P scale is a/h. Thus the P scale is a useful conversion agent. As an example and thought provoker, the hairline in Fig. 42 is shown in two positions, each of which shows that, if one side of a right triangle is 60 per cent of the hypotenuse, the other side will be 80 per cent of the hypotenuse.

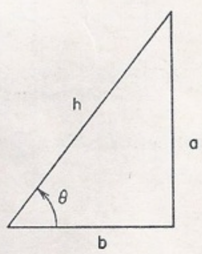

Fig. 41—Basic right triangle.

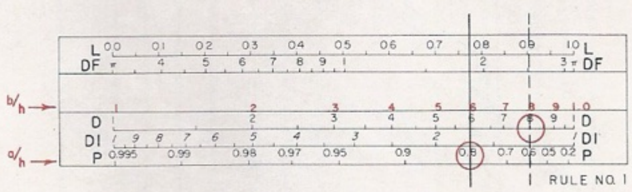

Fig. 42—Pythagorean scale use.

Trigonometry

If the sine, cosine and tangent functions are understood, the use of a slide rule to solve trigonometry problems is very simple. If these three functions are *not* understood, the slide rule becomes a device for aiding confusion. Most engineers first meet these functions in trigonometry and go through the rest of their careers plagued by a picture of a right triangle that acts as a mental block to developing a feeling for the uses of the sine and cosine functions in the study of harmonic motion, a-c circuits, radio waves, missile trajectories, etc. There is no "right" or "wrong" approach but the following development has been found by experiment to be very helpful and adaptable to all applications of periodic functions.

The sine (sin), cosine (cos) and exponential functions (e^z) are defined by the following infinite series:

$$\text{Sin } x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.$$

$$\text{Cos } x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots.$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots.$$

These series are good for all finite values of z and x: positive, negative, fractional, zero, etc. They are basic definitions; they have nothing to do with right triangles. Sin x and cos x are plotted in Fig. 43. Notice that they have maximum and minimum values of $\pm$ 1 and repeat themselves every 6.28 units (2π) of x. As can be seen from the curves, or proven by differentiation of their series, each is $\pm$ the plot of the slope of the other, a property which is unique to this pair of functions and is responsible for their occurrence in many physical problems of motion.

The number e can be determined by letting $z = 1$ in the expression for e^z.
In this case:

$$e = e^1 = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \ldots. = 2.71828 +$$

So e is a real live number, similar to π in that it cannot be expressed as the quotient of two whole numbers. The function e^z is plotted as the solid curve in Fig. 44. It is unique in that it is the only function whose value is identical to its slope at all points.

It is necessary to introduce a new number, j, at this point. We define j as follows:

$$j = \sqrt{-1} \quad (\text{or } j^2 = -1)$$

It can be shown that j obeys all the laws of algebra. The argument that follows will assume that it does. To see some of peculiarities of j notice that:

$$j^2 = -1, \quad j^3 = -j, \quad j^4 = 1, \quad j^5 = j, \quad 1/j = -j$$

(These can all be verified by substituting $\sqrt{-1}$ for j.) By letting $z = jx$ or $-jx$ in the series expression for e^z, the following series will result:

$$e^{jx} = 1 + jx - \frac{x^2}{2!} - \frac{jx^3}{3!} + \frac{x^4}{4!} + \ldots.$$

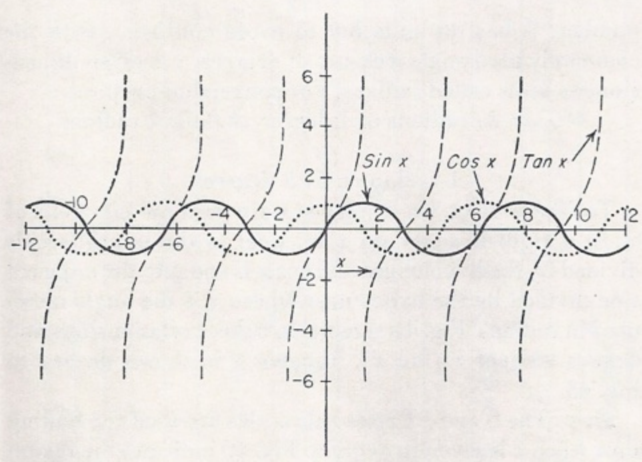

Fig. 43—Periodic functions.

$$e^{-jx} = 1 - jx - \frac{x^2}{2!} + \frac{jx^3}{3!} + \frac{x^4}{4!} + \ldots\ldots$$

By algebra it can be shown that the sin x and cos x series expression can be formed by combining the e^{jx} and e^{-jx} series so that for all values of x,

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

and

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

By taking the last two equations and combining them, the following equation results:

$$\sin^2 x + \cos^2 x = 1 \text{ (for all values of } x\text{)}$$

It is now possible to show that sin and cos have a trigonometry application. From plane geometry, the sides of the triangle of Fig. 41 have the following relationship:

$$a^2 + b^2 = h^2$$

or $$\frac{a^2}{h^2} + \frac{b^2}{h^2} = 1$$

or $$\left(\frac{a}{h}\right)^2 + \left(\frac{b}{h}\right)^2 = 1$$

which resembles $\sin^2 x + \cos^2 x = 1$.

As a result of this similarity of form, if the ratio of a side a to the hypotenuse, h, is called sin x, then the ratio of side b to the hypotenuse is given by cos x.

In symbol form,

$$\sin x = \frac{a}{h}$$

and $$\cos x = \frac{b}{h}$$

One question remains to complete our picture: What is x? Looking at Fig. 43, it is seen that, when x is zero, $\sin x = 0$ and $\cos x = 1$. As x increases from zero, sin x increases and cos x decreases. Fig. 41 shows that, if h is kept constant and the angle θ is increased from zero, then the growth of side a corresponds to sin x as has been demonstrated and the growth of θ is equivalent to a growth in x. It can be proven that x is a direct measure of the angle θ. x has a value of zero at $\theta = 0$ and a value of $\frac{\pi}{2}$ when θ is 90°. As x is a number, it has no units but to avoid confusion with the commonly used angle measure of degrees, x is given dimensionless units called radians. For conversion purposes:

$$360^\circ = 2\pi \text{ radians or } 1 \text{ degree} = 0.0174 \text{ radians}$$

or

$$1 \text{ radian} = 57.3 \text{ degrees.}$$

To recapitulate, sin x and cos x are periodic functions of x. In a right triangle sin x is equal to the opposite side divided by the hypotenuse and cos x is equal to the adjacent side divided by the hypotenuse where x is the angle measured in radians. Fig. 45 summarizes these relationships and defines tangent x (tan x). Tangent x is shown dashed in Fig. 43.

Sine. The S and ST slide rule scales are used for finding sin x when x is given in *degrees*. Fig. 46 indicates the region of the sine curve covered by the slide rule. One quarter of a cycle is tabulated on the ST and S scales and the other three quarters can be readily deduced when needed since the four quarters of the cycle have identical shapes.

Fig. 47 shows the scale layout. Sin x appears on the D scale under x in degrees set on the ST or S scale. For

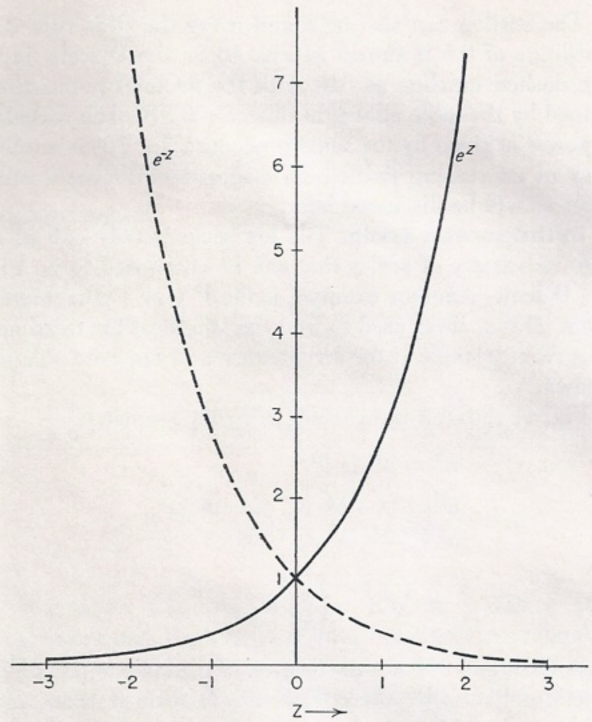

Fig. 44—Exponential functions.

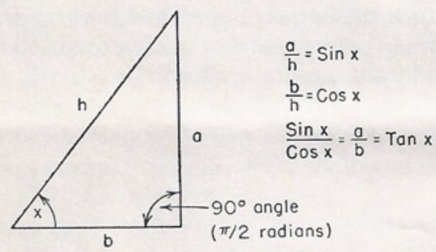

Fig. 45—Tangent x = Sin x/Cos x = a/b

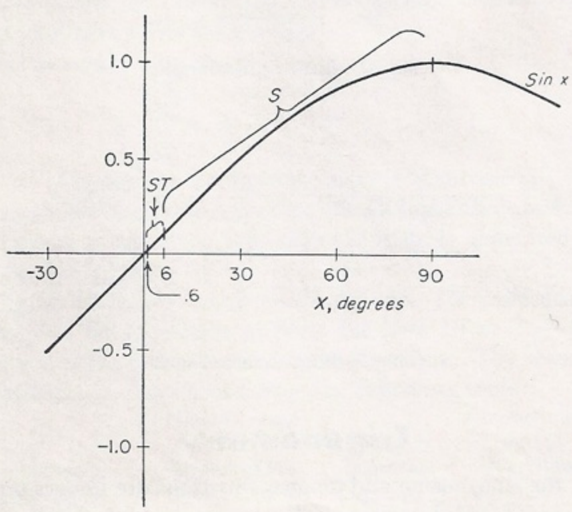

Fig. 46—Slide rule coverage of sine function.

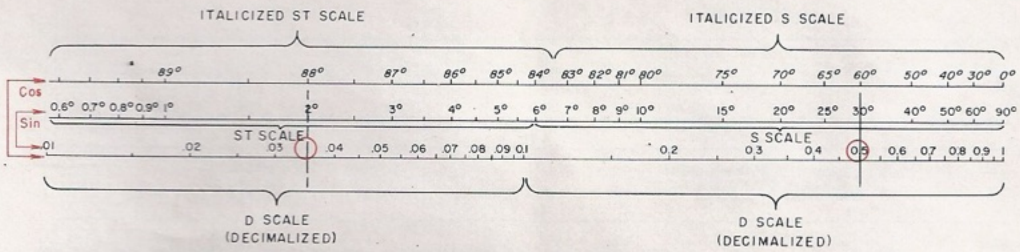

Fig. 47—Sine scale layout.

the dashed hairline indicates that sin 2° = 0.035. In Fig. 47 the D scales have been decimalized, but on the slide rule this insertion of the decimal point must be done by the user. Thus in Fig. 48 the dashed hairline shows the setting for sin 2° = 0.035 and the solid hairline shows the setting for sin 30° = 0.5.

For angles less than 0.01 radian (0.6 degrees) a very close approximation is given by

$$\sin x = x \ (x < .01)$$

where x is in radians. If the angle measure is given in degrees,

$$\sin y = 0.0174y \ (y < 0.6^\circ)$$

Thus for all angles the sine can be instantly converted to a distance on the D scale.

Since the distance from the left index represents the sine as indicated by the above discussion, referral to the D scale is not always necessary. For example, Fig. 49 solves:

$$3 \div (\sin 10^\circ) \times (\sin 3^\circ) = 0.905$$

[(···) − (−·−·) + (− − −) = (——)]

and Fig. 50 solves* $\sin^{-1}(3 \div 6)$. In this case $3 \div 6 = 0.5$ [(···) − (− − −) = (——)] is solved first and the answer, 30°, is found on the S scale opposite 0.5.

Cosine. Since the cosine is related to the sine by: $\sin(x) = \cos(90 - x)$, when x is in degrees, the cosine can be read from the ST and S curves by using the italicized numbers instead of the bold face numbers. (The italicized numbers are 90° minus the bold face numbers.) Thus Fig. 49 solves:

$$3 \div (\cos 80^\circ) \times (\cos 87^\circ) = 0.905$$

[(···) − (−·−·) + (− − −) = (——)]

Tangent. The T and ST** scales are used to find tan x when x is given in *degrees*. Fig. 51 indicates the region of the tangent curve covered by the slide rule. Almost one half of a cycle is tabulated on the T and ST scales and the other half of the tangent cycle can be readily determined by symmetry. Fig. 52 shows the scale layout. Tan x appears on the D or DI scale under x in degrees set on the T or ST scale. Eight examples are shown by different hairline settings in Fig. 52: tan 2° = 0.035, tan 4° = 0.07, tan 8½° = 0.15, tan 30° = 0.58, tan 65° = 2.14, tan 83° = 8.15, tan 85° = 11.5, tan 89° = 57.3. These examples should be carefully compared with the same tangent settings in Fig. 53 which shows how they appear on a slide rule. (In Fig. 53 the CI scale is used as the DI scale that appears on most large rules.) No elaborate memory tricks are necessary to tell whether the tangent of an angle appears on the D or DI scale, for it is only necessary to note that, if the angle is printed in bold type (under 45°), its tangent value is in bold type (D scale) and if the angle is printed in italics (over 45°), its tangent value is in italics (DI scale). In Fig. 52 the D and DI scales have been decimalized, but on

*Sin-1 means "angle whose sine is".

** The ST scale is used for both *Sine* and *Tangent* of small angles.

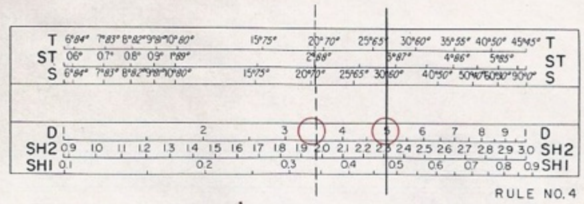

Fig. 48—Sin 2° = 0.035 and sin 30° = 0.5

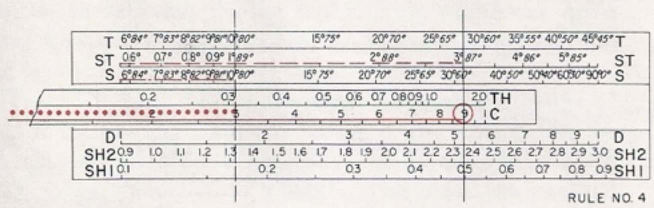

Fig. 49—3 ÷ sin 10° X sin 3° = 0.905

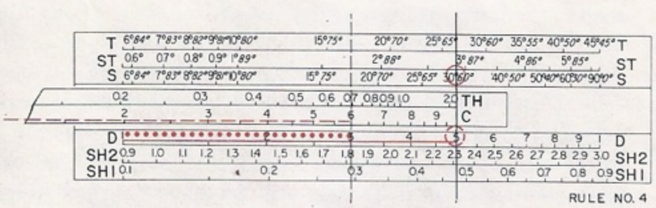

Fig. 50—$\sin^{-1}(3/6) = 30^\circ$

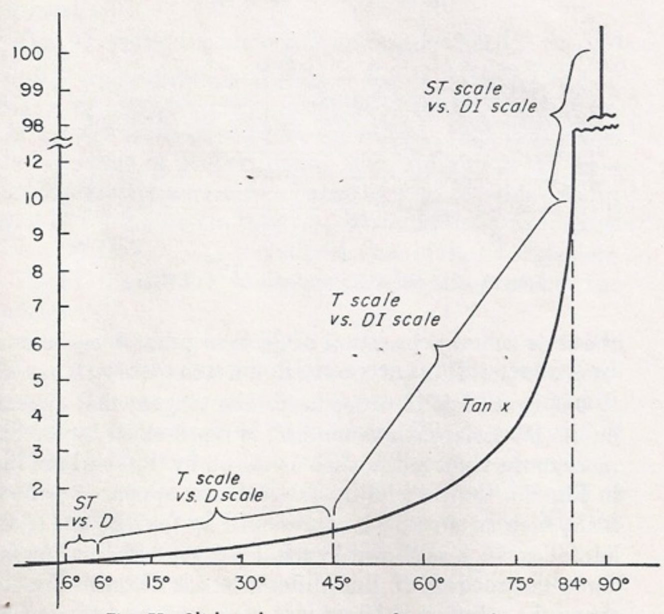

Fig. 51—Slide rule coverage of tangent function.

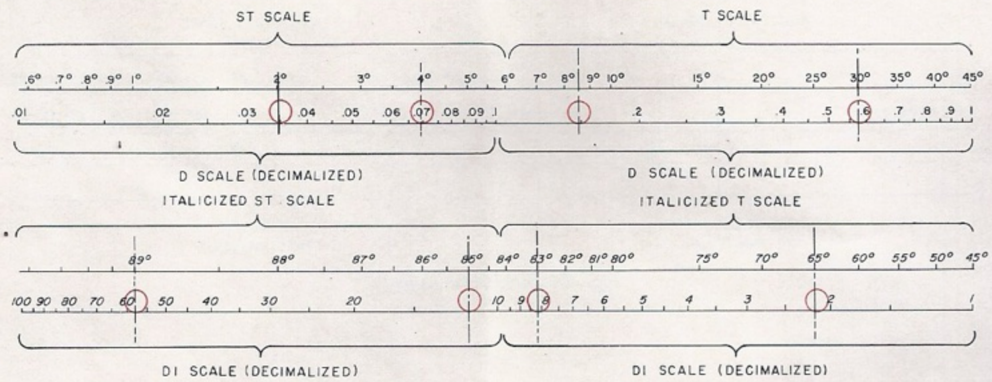

Fig. 52—Tangent scale layout.

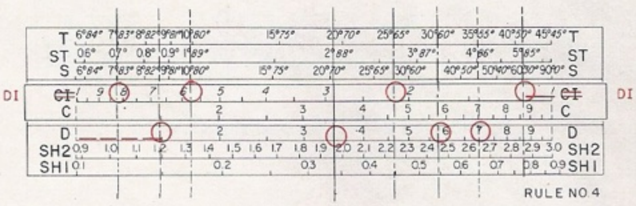

Fig. 53—Tangents of various angles.

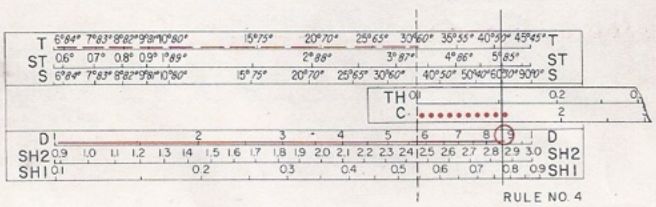

Fig. 54—15 X tan 30° = 8.66

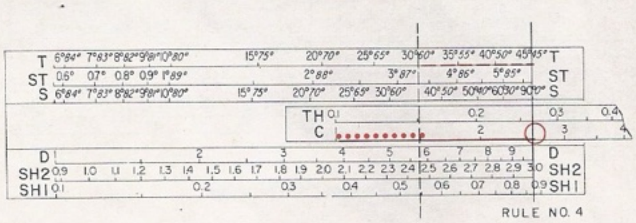

Fig. 55—15 X tan 60° = 26

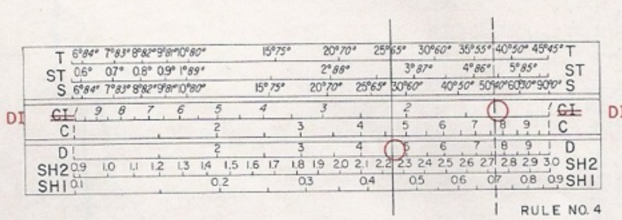

Fig. 56—Csc 50° = 1.3 and cot 65° = 0.466

the slide rule this insertion of decimal point must be done by the user (as was necessary in the sine case too).

One caution is in order here. The tangent that appears on the DI scale, such as tan 85°, is represented by the distance to the right index, such as shown by the solid red line in Fig. 53. Conversely, the tangent that appears on the D scale, such as tan 8.5°, is represented by the distance to the left index, such as shown by the dashed red line in Fig. 53. Two illustrations of this difference are shown. Fig. 54 shows the solution of 15 × (tan 30°) = 8.66 [(···) + (– – –) = (——)] and Fig. 55 shows 15 × (tan 60°) = 26 [(···) + (– – –) = (——)].

For angles less than 0.01 radians a very close approximation is given by

$$\tan x = x \ (x < 0.01)$$

where x is in radians. If the angle measure is given in degrees,

$$\tan y = 0.0174y \ (y < 0.6^\circ)$$

For angles over 89.4°, the tangent can be computed from

$$\tan (90^\circ - y) = \frac{1}{0.0174y} \ (y < 0.6^\circ)$$

where y is in degrees. For example,

$$\tan (89.5^\circ) = \tan (90^\circ - 0.5^\circ) = \frac{1}{0.0174 \times 0.5} = 115$$

Reciprocal Functions. Cosecant, secant and cotangent are easily read off the rule since they are reciprocals of sine, cosine and tangent respectively. So to find cosecant 50°, for example, find sin 50° and then read the reciprocal of sin 50° as csc 50°. Thus in Fig. 56, csc 50° = 1.3 is shown by the dashed hairline and cot 65° = .466 is shown by the solid hairline.

The following reference table permits easy conversion of functions of angles over 90° into functions of angles under 90°.

Angle lies in range	Function to be evaluated	Function is equal to
90° – 180°	$\sin x$	$\sin (180 - x)$
	$\cos x$	$-\cos (180 - x)$
	$\tan x$	$-\tan (180 - x)$
180° – 270°	$\sin x$	$-\sin (x - 180)$
	$\cos x$	$-\cos (x - 180)$
	$\tan x$	$\tan (x - 180)$
270° – 360°	$\sin x$	$-\sin (360 - x)$
	$\cos x$	$\cos (360 - x)$
	$\tan x$	$-\tan (360 - x)$

(where x is in degrees.)

The Log-Log Scales

The log-log scales on a slide rule are primarily designed to raise numbers to various powers, i.e., to evaluate such expressions as $7^{1.4}$, $8^{1.2}$, $83^{-2.5}$. Just as the C and D scales reduce the operation of multiplication to an addition of distances, the log-log scales reduce the operation of raising a number to a power to a simple multiplication which is then done on the C and D scales in the usual manner. The scheme used is as follows:

It is desired to calculate y^a (for example, 3^5).

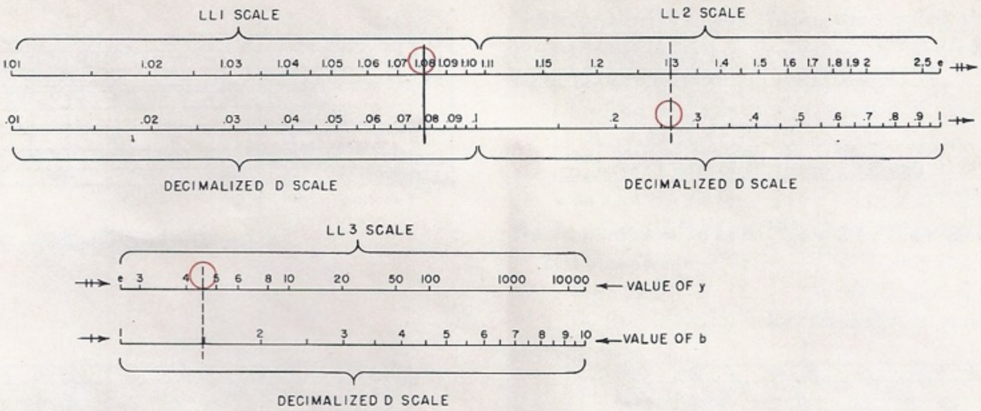

Fig. 57—Log-log scale arrangement for numbers greater than one.

1. The first step is to rewrite y as e^b (in this case $y = 3 = e^{1.1}$)*

2. The next step is to raise e^b to the a power by multiplying $b \times a$ to give e^{ba}. (In this case $(e^b)^a = e^{1.1 \times 5} = e^{5.5}$)

3. The final step is to evaluate e^{ba} to obtain the desired number. (In this case $e^{5.5} = 246 =$ answer).

Notice that the first step required a table to convert the number, y, to the form e^b; that is, to state b if given y. The second step was to multiply b by a and the third was to convert the answer from the form e^{ba} back to a number. Therefore, steps one and three require a conversion table and step two requires a multiplication device. The slide rule is already able to multiply, therefore only the conversion table referred to is necessary. This table is provided by the log-log scales which are arranged as shown in Fig. 57.

In Fig. 57 the numbers, y, from 1.01 to 10,000 appear on the LL1, LL2, LL3 scales. Directly below them appear the corresponding numbers of the exponent b such that $y = e^b$. The solid hairline in Fig. 57 shows that $1.08 = e^{0.077}$, the dashed hairline is set at $1.3 = e^{0.262}$ and the dotted hairline indicates that $e^{1.5} = 4.48$. This conversion chart is used for steps one and three referred to above. The D scale, when used with the C scale in the normal manner, will perform the multiplication of step two.

In the construction of the slide rule, the D scale is not decimalized as it is in Fig. 57 so that the user must provide the decimal point. The LL (log-log) scales appear above each other on the slide rule in a manner similar to the R scale arrangement. The same three problems worked out in Fig. 57, $1.08 = e^{0.077}$, $1.3 = e^{0.262}$ and $e^{1.5} = 4.48$ are shown in Fig. 58 under the solid, dashed, and dotted hairlines respectively for comparison with Fig. 57.

Raising A Number to a Power. Fig. 59A shows the solution of $(5)^{2.5} = 56$ or $\ln 5 \times 2.5 = \ln 56$ [(···) + (– – –) = (——)]. This notation will now be explained. The dashed hairline is placed over 5 on the LL3 scale. The reading on the D scale of 1.61 is the natural logarithm (abbreviated ln) of 5. In other words, $e^{1.61} = 5$. In order to raise 5 to the 2.5 power, it is necessary to multiply ln 5 by 2.5 ($1.61 \times 2.5 = 4$). Since the original number, 5, equalled $e^{1.61}$, the new number, $e^{1.6 \times 2.5} = e^4 =$ the original number raised to the 2.5 power. Glancing up the solid hairline e^4, as indicated by 4 on the D scale, is translated by the LL3 scale to its value of 56. Fig. 59B shows an alternate scheme for solving this problem using the CI scale. It is recommended that this problem and notation be reviewed carefully before proceeding.

Fig. 60 shows $(1.02)^{400} = 2{,}800$ or $\ln (1.02) \times 400 = \ln 2{,}800$ [(···) + (– – –) = (——)]. Notice that in this case the original number (1.02) and the final answer appear on different log-log scales. This will occur quite often and the reference to Fig. 57 will help straighten out the problem of scale selection.

The LL1, LL2, LL3 scales cover the number range of 1.01 to 20,000. Fig. 61 shows the LL01 and LL02 scales for numbers from 0.99 to 0.40. Many slide rules have an LL03 scale to go from 0.40 to 0.00005. This scale was omitted here to save space. As in the case of the other log-log scales, e raised to a power shown on the D scale is equal to the number indicated directly below on the LL scales. Notice that the D scale is labeled with negative numbers. These minus signs and the decimalization shown do not appear on the slide rule. A few simple examples show that the operation of the scales of Fig. 61 is the same as those of Fig. 57. Compare Fig. 62 with 61. Fig. 62 shows the slide rule layout for the same problem as set up in Fig. 61 and again points out the need for the user to insert decimals. The problems set up in these two figures are as follows: $e^{-0.5} = 0.607$ (dashed hairline), $e^{-0.035} = 0.9656$ (dotted hairline), $0.98 = e^{0.0202}$ (solid hairline), $0.93 = e^{-0.0725}$ (dash-dot hairline).

Fig. 63 shows the slide rule setting for $(0.8)^2 = 0.64$ using the log-log scales $(\ln 0.8) \times 2 = (\ln 0.64)$ [(···) + (– – –) = (——)]. Notice that the use of the LL01, LL02 scales is the same as the use of LL1, LL2, LL3 scales. The selection of proper log-log scale represents the same sort of problem here also as seen by Fig. 64 which shows $(0.98)^{45} = 0.4$ or $(\ln 0.98) \times 45 = \ln 0.4$ [(···) + (– – –) = (——)]. Notice the use of the CI scale here.

The LL01 and LL02 scales are italicized because they run backwards.

Because the LL1 and LL01 scales represent e raised to the same power at each hairline setting with one the negative and the other the positive exponent, the LL1 and LL01 scales are reciprocals of each other. This is also true of LL2 and LL02 scales. As a result, reciprocals of numbers near unity can be read off with great accuracy because as unity is approached these scales are very wide. Fig. 65 shows the slide rule setting for $0.975 = 1/1.026$.

On some slide rules, to save space, the log-log scales are half length and used with the A scale.

*$e = 2.72+$, a positive real number

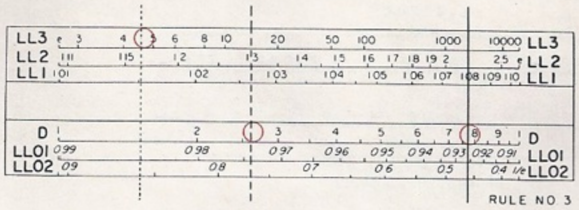

Fig. 58—$1.08 = e^{0.077}$; $1.3 = e^{0.262}$; and $e^{1.5} = 4.48$

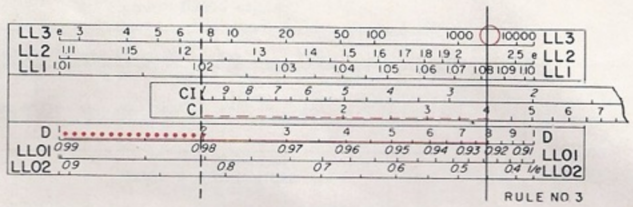

Fig. 60—$(1.02)^{400} = 2800$

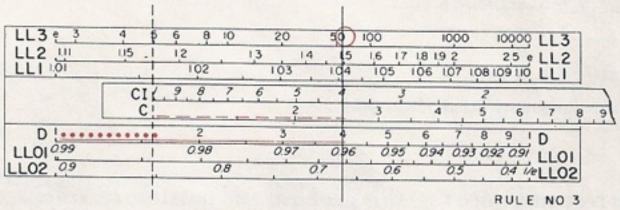

Fig. 59A—$(5)^{2.5} = 56$

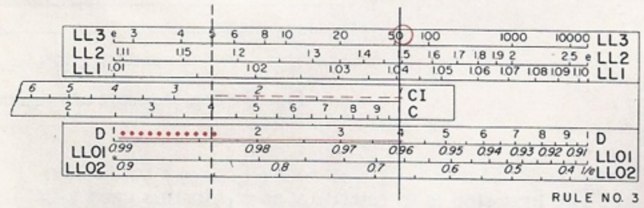

Fig. 59B—$(5)^{2.5} = 56$

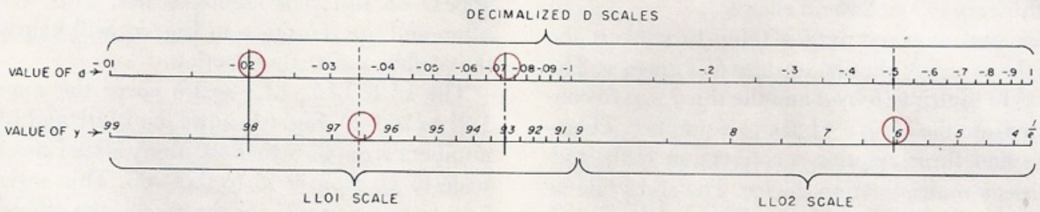

Fig. 61—Log-log scale arrangement for numbers less than one.

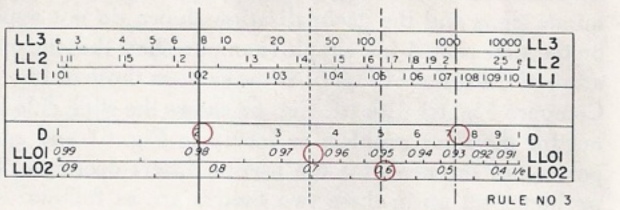

Fig. 62—$e^{-0.5} = 0.607$; $e^{-0.035} = 0.9656$; $0.98 = e^{-0.0202}$; and $0.93 = e^{-0.0725}$

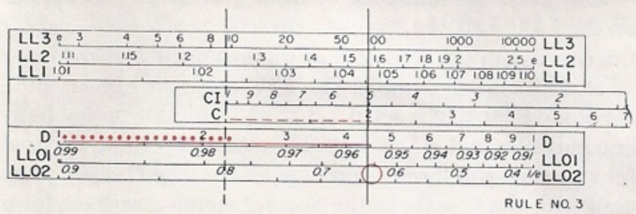

Fig. 63—$(0.8)^2 = 0.64$

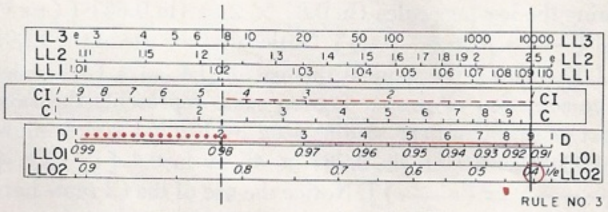

Fig. 64—$(0.98)^{45} = 0.4$

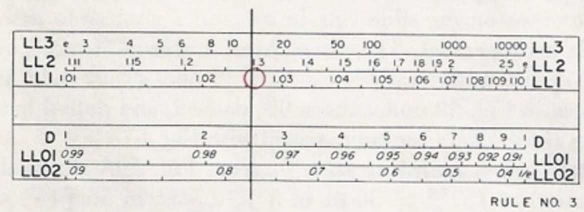

Fig. 65—$0.975 = 1/1.026$

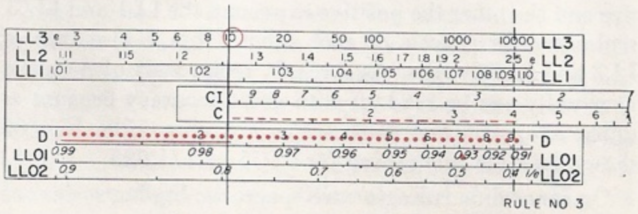

Fig. 66A—$(10{,}000)^{1/4} = 10$

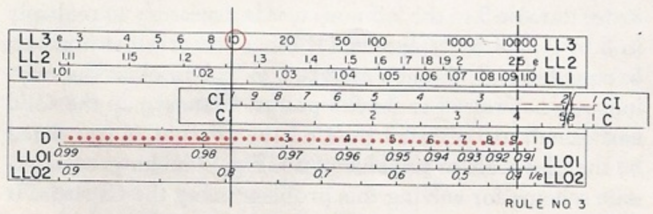

Fig. 66B—$(10{,}000)^{0.25} = 10$

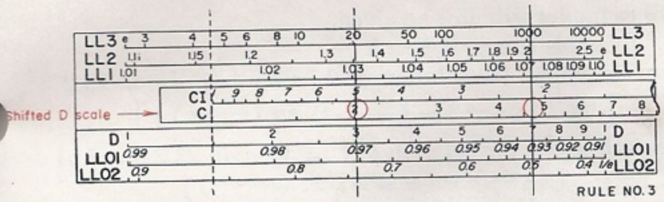

Fig. 67—Logarithms to the base 4.5

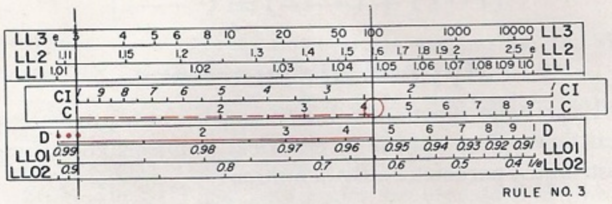

Fig. 68—$(3)^{4.2} = 100$

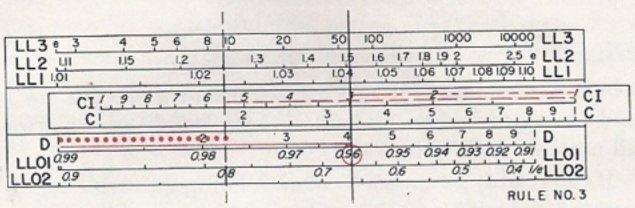

Fig. 69—$(0.8)^{5.5/30} = 0.96$

The log-log scales are based on powers of e. Any other number could have been chosen as a base, but e is commonly used since calculation of powers of e is a frequent chore.

Fractional Powers or Roots. Raising a number to a fractional power or root is also possible. Fig. 66A shows $(10{,}000)^{1/4} = 10$. $(\ln 10{,}000) \div 4 = \ln 10$ [(···) − (– – –) = (——)]. Fig. 66B shows another version of the same problem, $(10{,}000)^{0.25} = 10$ [$(\ln 10{,}000) \times 0.25 = \ln 10$] [(···) + (– – –) = (——)].

Odd Logarithms. The log-log scales permit calculating logarithms to any base by shifting the D scale (this is done by using the C scale as a substitute) so that the shifted D scale index is in line with the desired base number on the log-log scale. In Fig. 67, the shifted D scale (C scale) has its index at 4.5 on scale LL3 as shown by the dotted hairline. As a result the dashed hairline shows that $\log_{4.5} 20 = 2$. The solid hairline shows $\log_{4.5} 1000 = 4.6$ and $\log_{4.5} 2 = 0.46$.

Exponential Equations. Fig. 68 illustrates the use of the slide rule in solving exponential equations. Here $3^y = 100$ is solved for y. The procedure is to set up the solution as if y is known and the answer, 100, is not known. The last step, the setting of the solid hairline, is then performed by setting the hairline (on y) on the C scale so that the hairline covers 100 on the LL3 scale. The value of y is then read as 4.2 as shown. The solution of Fig. 68 follows: [$(\ln 3) \times y = \ln 100$] [(···) + (– – –) = (——)].

Raising a number to a power and extracting a root is a combination of previous operations such as shown in Fig. 69 for the case: $(0.8)^{5.5/30} = 0.96$ [$(\ln 0.8) \times 5.5 \div 30 = 0.96$] [(···) + (– – –) − (–·–·) = (——)].

Extended Values. Since the log-log scales have finite limits, occasionally a problem will "fall off the end of the rule." There are four possible cases:

1. To find $\ln y$ if y lies between 0.99 and 1.01, rewrite y as: $(1 + z)$ or $(1 - z)$.

$$\ln(1+z) = z \quad (z < 0.01)$$
$$\ln(1-z) = -z \quad (z < 0.01)$$

As an example, $\ln 1.002 = \ln(1 + 0.002) = 0.002$.

2. To evaluate e^n if $|n| < 0.01$, use the approximation:

$$e^n = 1 + n \qquad (-0.01 < n < +0.01)$$

3. To find $\ln y$ if y is less than 0.4 or greater than 10,000 rewrite $\ln y$ as

$$\ln y = \ln a + \ln b$$

where a and b are two numbers such that

$$y = a \times b,$$

selecting a and b such that they appear on the log-log scales.

4. To evaluate y^n if it is larger than 10,000 or smaller than 0.04 rewrite y^n as

$$y^n = a^n \times b^n$$

where a and b are two numbers such that a^n and b^n both appear on the log-log scale and are related by:

$$y = a \times b$$

Vector Diagrams

A very common type of problem is to resolve a vector* into components along perpendicular axes or to calculate the vector if its components are given. The electrical phasor diagram is a relative of the vector diagram and uses the same calculation techniques.

Fig. 70 shows the development of vector triangle relationships. Fig. 70A shows the basic or unit triangle – a right triangle with a hypotenuse of unit length. The sides are $\sin\theta$ and $\cos\theta$ in length where θ is the angle opposite the $\sin\theta$ side as described previously. Fig. 70B shows a vector quantity of magnitude r and direction angle θ which can be considered to be the sum of two components, b and a, in the vertical and horizontal directions, respec-

*A vector is a quantity, such as force, which has magnitude and direction as opposed to a scalar, such as temperature, which has only magnitude.

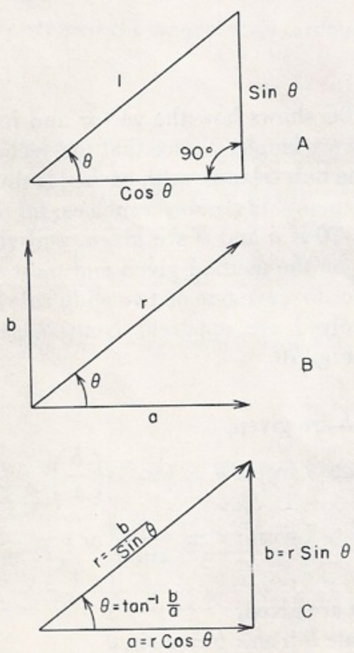

Fig. 70—Development of vector components.

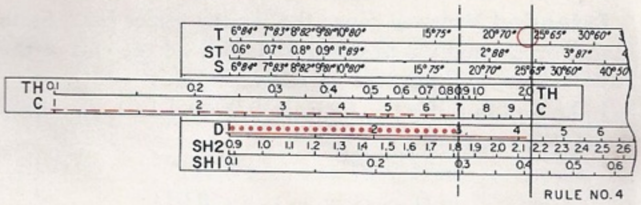

Fig. 71A—$\tan^{-1}(3/7) = 23.2°$

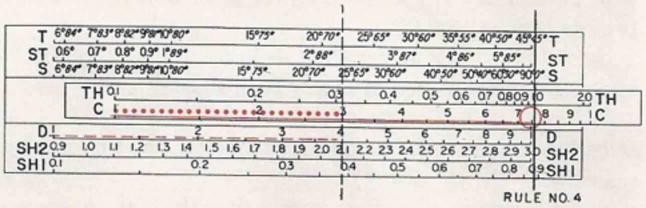

Fig. 71B—$3 \div \sin 23.2° = 7.61$

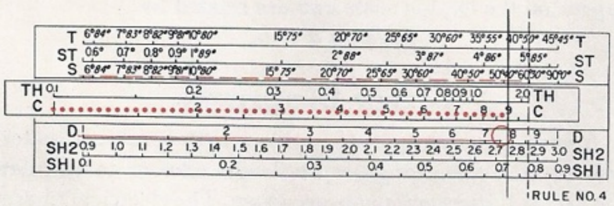

Fig. 72A—$9 \cos 30° = 7.8$

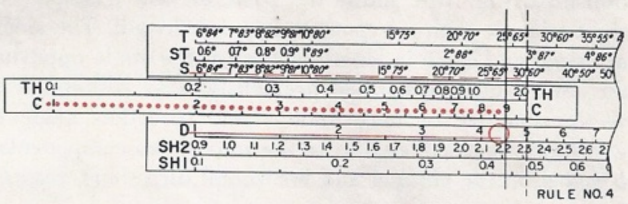

Fig. 72B—$9 \sin 30° = 4.5$

tively. Fig. 70C shows how the vector and its components are related in a triangle. Notice that the vector diagram of Fig. 70C is the unit triangle with each side multiplied by r.

There are many ingenious schemes for calculating r and θ of Fig. 70 if a and b are given, and vice versa, but all are based on the method given and most sacrifice simplicity in order to save one or two slide rule motions. The formulas involved are apparent from inspection of the diagrams of Fig. 70.

Method I:

If a and b are given,

1. Calculate θ from: $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

2. Calculate r from: $r = \frac{b}{\sin\theta}$ or $r = \frac{a}{\cos\theta}$

Method II:

If r and θ are given,

1. Calculate b from: $b = r \sin\theta$
2. Calculate a from: $a = r \cos\theta$

Two examples will now be given.

Example I: $a = 7$, $b = 3$

Fig. 71A shows the solution of $\theta = \tan^{-1}(3/7) = 23.2°$

[tan⁻¹ [(···) − (- - -) = (——)]]

Fig. 71B shows the solution of $r = 3 \div \sin 23.2° = 7.61$

[(···) − (- - -) = (——)]

Example II: $r = 9$, $\theta = 30°$

In Fig. 72A: $a = 9 \cos 30° = 7.8$

[(···) + (- - -) = (——)]

In Fig. 72B: $b = 9 \sin 30° = 4.5$

[(···) + (- - -) = (——)]

Hyperbolic Functions

Hyperbolic functions are first cousins to trigonometric functions. They appear in electrical problems containing distributed parameters. By definition:

hyperbolic sine of $x = \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \cdots\cdots,$

hyperbolic cosine of $x = \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \cdots,$

and

hyperbolic tangent of $x = \tanh x = \frac{\sinh x}{\cosh x},$

which are similar to the series definitions given for $\sin x$ and $\cos x$ previously. As before, these definitions hold for all possible finite values of x.

It can readily be shown by manipulating the series definitions of e^x, $\cosh x$ and $\sinh x$ that:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

and by combining these equations,

$$\cosh^2 x - \sinh^2 x = 1$$

It can be demonstrated that sinh and cosh are related to a triangle connected to a hyperbola just as sine and cosine are related to a triangle connected to a circle, but the analog seems to have almost no interpretative value in engineering

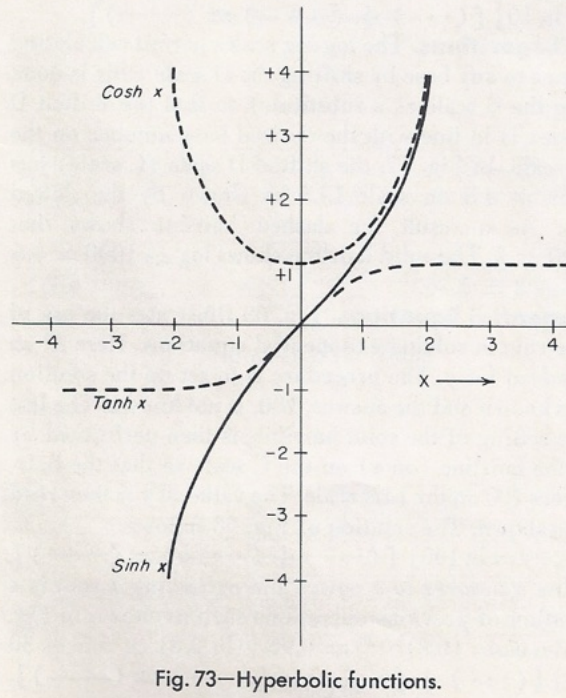

Fig. 73—Hyperbolic functions.

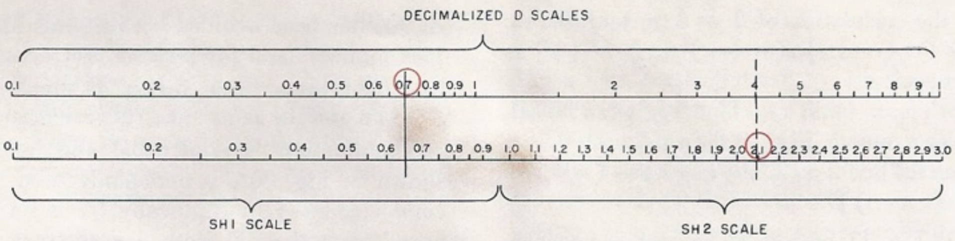

Fig. 74—Hyperbolic sine scale arrangement.

problems. This analog, however, is responsible for the names "hyperbolic sine" and "hyperbolic cosine," etc.

Fig. 73 shows plots of sinh x, cosh x and tanh x. Notice that, unlike trigonometric functions of Fig. 43, the hyperbolic functions are not periodic.

Sinh. The SH1 and SH2 scales are used to find sinh x when x is given in radians. Fig. 74 shows the scale layout. Sinh x appears on the D scale over x in radians set on the SH1 or SH2 scale. For example, the solid hairline shows that sinh 0.65 = 0.696 and the dashed hairline indicates that $\sinh^{-1} 4 = 2.1$ radians. In Fig. 74 the D scales have been decimalized, but on the slide rule the user must insert the decimal point, as usual. Thus, in Fig. 75 the solid hairline shows that sinh 0.65 = 0.696 and that sinh 2.64 = 7. The dashed hairline in Fig. 75 indicates that $\sinh^{-1} 4 = 2.1$ radians. Compare Figs. 74 and 75.

For small values of x beyond the SH scale, the following approximation can be used:

$$\sinh x = x \quad (x < 0.1) \quad (x \text{ is in radians})$$

For values of x larger than 3.0, the following approximation is useful:

$$\sinh x = \frac{e^x}{2} \quad (x > 3.0) \quad (x \text{ is in radians})$$

The evaluation of e^x is performed using the log-log scales as described previously.

Notice that the distance from the left index to the value of x on either SH scale represents sinh x in D scale units. This is illustrated in the following example of SH scale use. Fig. 76 solves:

3 ÷ (sinh 2) × (sinh 0.7) = 0.63 [(• • •) − (− − −) + (−•−•) = (——)]

Should it be necessary to evaluate sinh of a negative number, as can be seen in Fig. 73, sinh $(-x) = -\sinh x$.

Tanh. The TH scale is used to find tanh x when x is given in radians. Fig. 77 shows the scale layout. Tanh x appears on the C scale under x in radians set on the TH scale. For example, the dashed hairline shows that tanh 0.8 = 0.665. Conversely, the solid hairline shows that $\tanh^{-1} 0.5 = 0.55$ radians. In Fig. 77 the C scale has been decimalized, but on the slide rule the user must provide the decimal point.

Tanh x is always less than one. As a result, the tanh scale on the slide rule gives all values of tanh x for the range $x = 0.01$ to $x = \infty$. For values of $x < 0.01$, the following approximation can be used:

$$\tanh x = x \quad (x < 0.01) \quad (x \text{ is in radians})$$

Notice that the distance from the left index to the value of x on the TH scale represents tanh x in C scale units. If x is negative the same scale is used and the following applies: $-\tanh x = \tanh(-x)$, as can be verified by examination of Fig. 73.

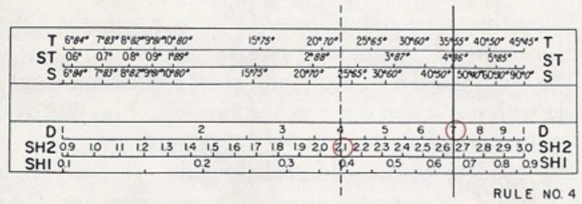

Fig. 75—Sinh 0.65 = 0.696 and sinh 2.64 = 7

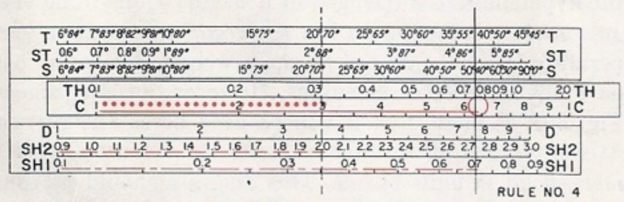

Fig. 76—3 ÷ (sinh 2) X (sinh 0.7) = 0.63

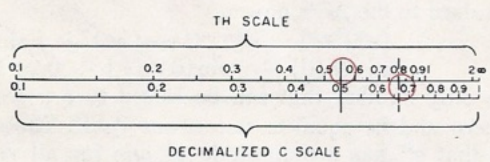

Fig. 77—Hyperbolic tangent scale arrangement.

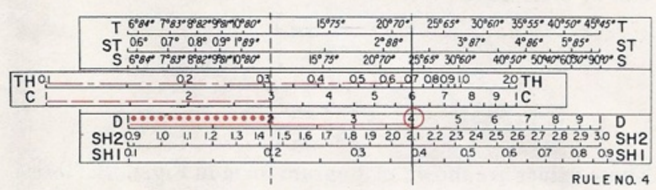

Fig. 78—2 ÷ 3 X (tanh 0.7) = 0.403

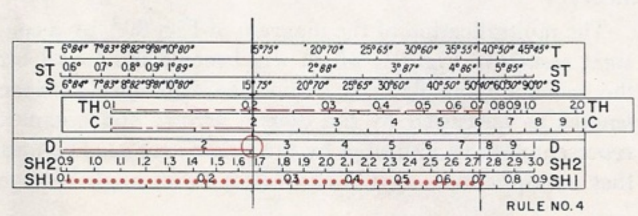

Fig. 79—2 X (cosh 0.7) = (sinh 0.7) ÷ (tanh 0.7) X 2 = 2.52

Fig. 78 shows the calculation of $2 \div 3 \times \tanh 0.7 = 0.403$ [(· · ·) − (− − −) + (−·−·) = (——)] as an example of TH scale use.

Cosh. Since $\cosh x = (\sinh x)/(\tanh x)$, it can be calculated readily. For example, Fig. 79 shows: $2 \cosh 0.7 = (\sinh 0.7) \div (\tanh 0.7) \times 2 = 2.52$ [(· · ·) − (− − −) + (−·−·) = (——)]

As can be seen in Fig. 73, $\cosh (-x) = \cosh x$, i.e. cosh is always positive and is the same for either positive or negative values of a given x.

If x is less than 0.1 radians, $\cosh x$ is unity for all practical purposes.

Phasor Calculations

From the series definitions of e^z, $\cos x$ and $\sin x$ it can be shown that $e^{jx} = \cos x + j\sin x$.

The combination of $\cos x$ and $\sin x$ suggests a right triangle with $\cos x$ and $\sin x$ as the two legs. Fig. 80A shows a diagram *which is only an aid to calculations* and is commonly used to picture this relationship. This diagram is *not* a proof or demonstration of a real meaning for e^{jx}. e^{jx} is a number—nothing more. The diagram of Fig. 80A may be thought of as an analog or an aid to computation, but it should not be assumed that it gives e^{jx} a meaning as the hypotenuse of a triangle, or a vector, or anything else. Just as the number seven can represent a distance, a temperature, a time, a size of group, a weight, etc., so e^{jx} can have any number of meanings. However, the diagram of Fig. 80A is handy; so it will be used here. The vertical axis is marked off in units of j and the horizontal axis is marked off in units of one. This diagram should be compared with Fig. 70 which it resembles closely.

Since $\sin^2 x + \cos^2 x = 1$, the length of the hypotenuse (the magnitude of e^{jx}) in Fig. 80 is $\sin^2 x + \cos^2 x = 1$ for any value of x*. This checks with the observation that, if e^{jx} is raised to the $2\pi/x$ power,

$$(e^{jx})^{2\pi/x} = e^{j2\pi} = \cos(2\pi) + j\sin(2\pi)$$
$$= \cos(0) + j\sin(0) = 1 + 0 = 1$$

The only number that can be raised to a power other than zero and be equal to one is one itself. Therefore, it seems that e^{jx} has a magnitude of one for all values x! As x varies then, the e^{jx} line in Fig. 80A rotates about the origin shown with a constant line length of unity. In fact, by substitution into the formula, $e^{jx} = \cos x + j \sin x$, it can be seen that:

$$e^{o} = 1$$
$$e^{j\pi/2} = j$$
$$e^{j\pi} = -1$$
$$e^{j3\pi/2} = -j$$
$$e^{j2\pi} = 1$$

These values are shown in diagram form in Fig. 80B. Notice that e^{jx} rotates as x increases. By letting $x = jwt$ where w is frequency in radians per second and t is time in seconds, $e^{jx} = e^{jwt}$ becomes the unit phasor of a-c circuit theory.

The multiplication of the diagram of Fig. 80A by a constant results in Fig. 81, which will become the model for the following calculations. Notice the difference between lengths as measured by the double arrows and complex representation as indicated by the single arrows. The idea that multiplication by j is equivalent to an angle change of 90° has been avoided because this idea holds for complex numbers and for phasors, but does not hold for vectors; yet these three forms of numbers use the same diagram and the same slide rule calculation!

The notation "axis of reals" and "axis of imaginaries" shown in Fig. 80A is commonly used and is shown for completeness. Philosophically, $j7$ is no more an "imaginary" term than 7. Both are abstract symbols that are aids to logical manipulation.

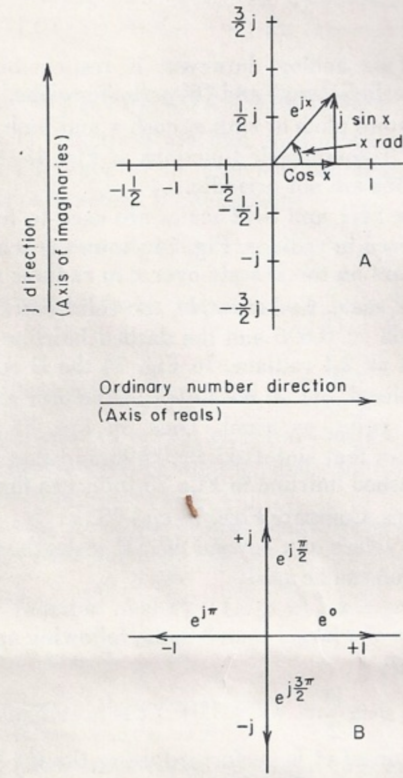

Fig. 80—Representation of e^{jx}.

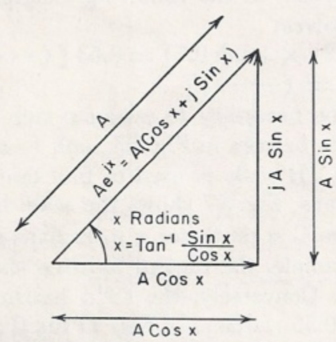

Fig. 81—Complex term evaluation.

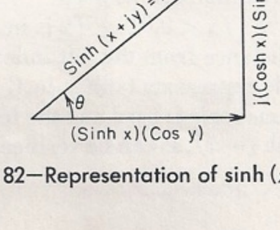

Fig. 82—Representation of $\sinh (x + jy)$

*Note that although the vertical side of the triangle is $j \sin x$, the length of that side is plain "sin x."

Evaluation of Complex Functions. A method for the evaluation of $\sinh(x + jy)$, $\sin(x + jy)$, etc., will be given based on the diagram of Fig. 81. There are more flashy methods, but the approach shown is universal, consistent and easy to learn. As in many fields, once a simple basic approach is mastered, tricks can be readily picked up as needed. The method will be applied to $\sinh(x + jy)$ as an example.

It can be shown by substitution in the various series expansions that:

$$\sinh(x + jy) = (\sinh x)(\cos y) + j(\cosh x)(\sin y)$$

or

$$\sinh(x+jy) = (\sinh x)(\cos y) + j(\sinh x)(\sin y)/(\tanh x)^*$$

This relationship is represented in Fig. 82. From this Fig.,

$$\sinh(x + jy) = Ae^{j\theta}$$

where $\theta = \tan^{-1}\dfrac{\cosh x \sin y}{\sinh x \cos y} = \tan^{-1}\dfrac{\tan y^*}{\tanh x}$

and $A = \dfrac{\sinh x \cos y^*}{\cos\theta}$

The forms with asterisks avoid $\cosh x$ and are therefore readily used for slide rule computation. Note that the slide rule version of the θ equation is formed by replacing cosh by sinh/tanh since cosh is not a slide rule scale. Essentially, the method used is to draw a right triangle representation of the original cartesian expression. The polar representation is easily deduced from that diagram using the vector techniques developed previously.

An example of computing $\sinh(2 + j1.22)$ in cartesian form will point out a difficulty. From the formula given:

$$\sinh(2 + j1.22) =$$
$$(\sinh 2)(\cos 1.22) + j(\sinh 2)(\sin 1.22)/(\tanh 2)$$

but cos 1.22 and sin 1.22 must be converted into degrees to use the slide rule scales. Since

$$1.22 \text{ radians} = 1.22 \times 57.3^\circ = 70^\circ,$$
$$\sinh(2 + j1.22) =$$
$$(\sinh 2)(\cos 70^\circ) + j(\sinh 2)(\sin 70^\circ)/(\tanh 2)$$

It is suggested that the reader calculate this on his slide rule. The answer is:

$$\sinh(2+j1.22) = (3.62)\times(.342) + j(3.62)(.94)/(.965)$$
$$= 1.24 + j3.52$$

The cartesian forms of other complex functions are:

$$\cosh(x + jy) = (\cosh x)(\cos y) + j(\sinh x)(\sin y)$$
$$\sin(x + jy) = (\sin x)(\cosh y) + j(\cos x)(\sinh y)$$
$$\cos(x + jy) = (\cos x)(\cosh y) - j(\sin x)(\sinh y)$$

They can be evaluated by the same scheme used for $\sinh(x + jy)$.

For computing $\tan(x - jy)$ or $\tanh(x - jy)$, because of the extreme values that often occur, it is usually best to first evaluate sinh and cosh or sin and cos and then substitute in either tanh = sinh/cosh or tan = sin/cos as appropriate.

Circular Slide Rules

The circular slide rule is almost as old as the straight slide rule. A sketch of one is shown in Fig. 83. The three scales shown, C, CI, A, are just a few of the usual array which often includes more scales than the most elaborate straight slide rule.

In principle, the operation of a circular slide rule is the same as the operation of a straight slide rule. Distances

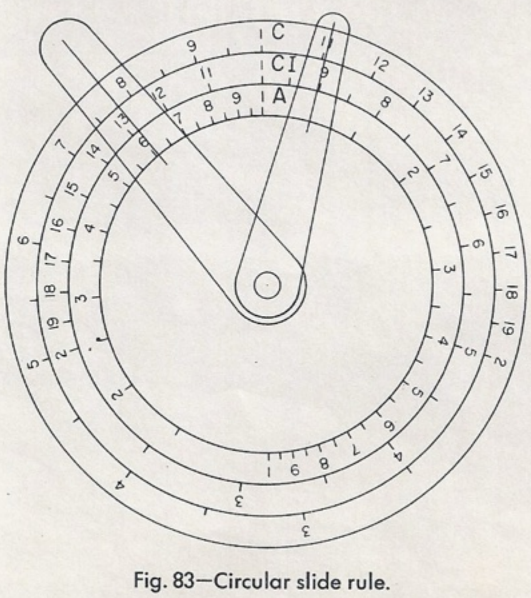

Fig. 83—Circular slide rule.

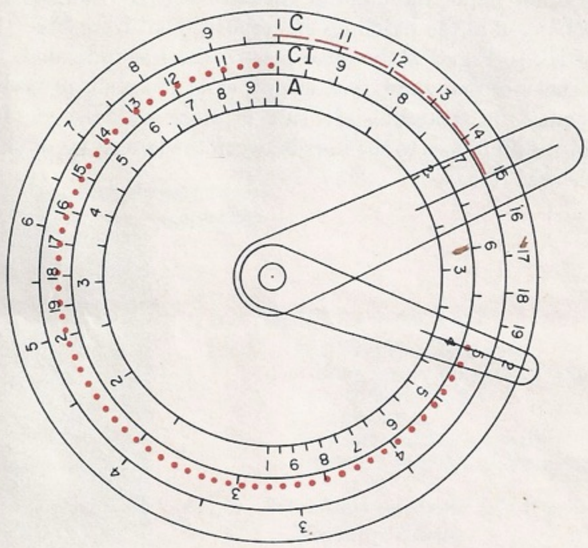

Fig. 84A—5 X 1.5

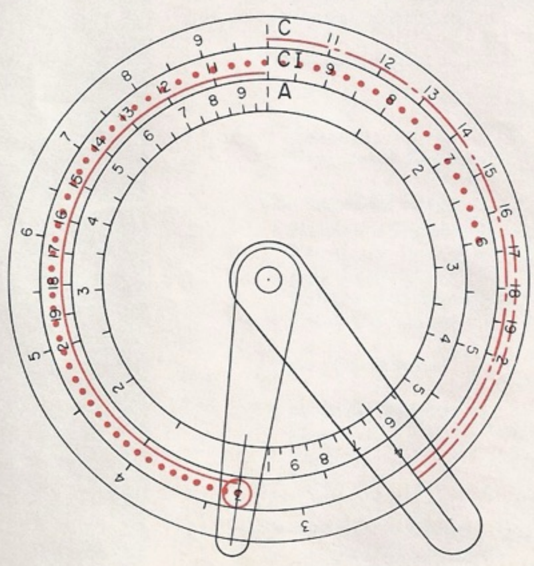

Fig. 84B—5 X 1.5 ÷ 2½ = 3

are added to correspond to multiplying and subtracted to correspond to division. The two plastic fingers which contain the hairlines are free to pivot about the center and are mounted so that if one hairline is left free and the second is rotated, the first hairline moves with the second so that the angle between the two hairlines remains fixed. If it is desired to change the angle between the hairlines, one is grasped in each hand and the angle is varied by causing a friction clutch that connects the two hairlines to slip.

An example of circular slide rule use is shown in Figs. 84A and 84B in solving the problem

$5 \times 1.5 \div 2.5 = 3$

[(···) + (– – –) – (–·–·) = (——)]

In Fig. 84A, 5×1.5 [(···) + (– – –)] is formed as a distance by putting one hairline on 5 on the CI scale and the other hairline on 1.5 on the C scale. The angle between now corresponds to 5×1.5. (Notice that in the case of the circular slide rule, angles correspond to the distances on a straight slide rule). By holding the circular rule in one hand and moving one of the hairlines with the other hand, the angular distance between hairlines is unchanged as the hairlines are repositioned from Fig. 84A to 84B. In Fig. 84B the answer is read off as indicated.

The circular slide rule is not very popular, probably because the straight slide rule is more established. The major differences to the user between the two styles of slide rule are:

1. A 10 in. diameter circular rule has a C scale over 30 in. long and therefore can be read more accurately than a 10 in. straight slide rule.
2. The circular slide rule is cheaper to manufacture than a straight slide rule and can take more abuse.
3. Because the scales are continuous, i.e., one index touches the other, changing indices is never necessary when using a circular slide rule as it is when using a straight slide rule.
4. The circular shape is more awkward to handle and carry than the straight shape.
5. Locating specific values on a scale is more difficult on a circular scale than a straight scale because at any given time half the numbers are upside down and the indices may be in any orientation.

Summary

As a summary and review of the basic relationships among the scales of a slide rule:

If a number x is placed on the C or D scales and the C and D indices are aligned, the other scales read as follows: LLO $= e^{-x}$, LL $= e^{x}$, DF and CF $= \pi x$, DI and CI $= x^{-1}$, T $= \tan^{-1}x$, S $= \sin^{-1}x$, K $= x^3$, A and B $= x^2$, L $=$ mantissa of $\log_{10}x$, TH $= \tanh^{-1}x$, SH $= \sinh^{-1}x$, P $= \sqrt{(1-x^2)}$, R $= x^{1/2}$.

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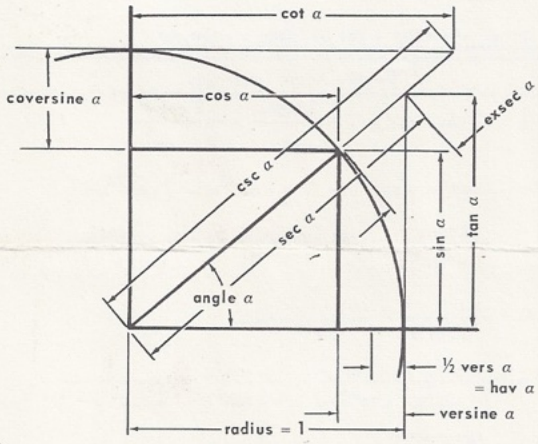
Trigonometric Reference Table

George D. Pheil, P. E., Racine, Wis.



	Ratio	Reciprocal	Product
$\sin\alpha =$	$\cos\alpha / \cot\alpha = \tan\alpha / \sec\alpha$	$= 1/\csc\alpha$	$= \cos\alpha \tan\alpha$
$\cos\alpha =$	$\sin\alpha / \tan\alpha = \cot\alpha / \csc\alpha$	$= 1/\sec\alpha$	$= \sin\alpha \cot\alpha$
$\tan\alpha =$	$\sin\alpha / \cos\alpha = \sec\alpha / \csc\alpha$	$= 1/\cot\alpha$	$= \sin\alpha \sec\alpha$
$\cot\alpha =$	$\cos\alpha / \sin\alpha = \csc\alpha / \sec\alpha$	$= 1/\tan\alpha$	$= \cos\alpha \csc\alpha$
$\sec\alpha =$	$\tan\alpha / \sin\alpha = \csc\alpha / \cot\alpha$	$= 1/\cos\alpha$	$= \tan\alpha \csc\alpha$
$\csc\alpha =$	$\cot\alpha / \cos\alpha = \sec\alpha / \tan\alpha$	$= 1/\sin\alpha$	$= \cot\alpha \sec\alpha$

Pythagorean

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \tan\alpha / \sqrt{1+\tan^2\alpha} = 1 / \sqrt{1+\cot^2\alpha}$$

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = 1 / \sqrt{1+\tan^2\alpha} = \cot\alpha / \sqrt{1+\cot^2\alpha}$$

$$\tan\alpha = \sqrt{\sec^2\alpha - 1}$$

$$\cot\alpha = \sqrt{\csc^2\alpha - 1}$$

$$\sec\alpha = \sqrt{1+\tan^2\alpha}$$

$$\csc\alpha = \sqrt{1+\cot^2\alpha}$$

Basic Trigonometric Relations with Unit Circle		Functions of $(-\alpha)$
$\sin\alpha$ = opp/hyp = opp/1 = opposite	versine α = vers α = $1-\cos\alpha$	$\sin(-\alpha) = -\sin\alpha$
$\cos\alpha$ = adj/hyp = adj/1 = adjacent	coversine α = covers α = $1-\sin\alpha$	$\cos(-\alpha) = \cos\alpha$
$\tan\alpha$ = opp/adj = opp/1 = opposite	haversine α = hav α = 1/2 vers α	$\tan(-\alpha) = -\tan\alpha$
$\cot\alpha$ = adj/opp = adj/1 = adjacent	exsecant α = exsec α = $\sec\alpha - 1$	$\cot(-\alpha) = -\cot\alpha$
$\sec\alpha$ = hyp/adj = hyp/1 = hypotenuse		$\sec(-\alpha) = \sec\alpha$
$\csc\alpha$ = hyp/opp = hyp/1 = hypotenuse		$\csc(-\alpha) = -\csc\alpha$

Cofunctions of $(\alpha \pm \pi/2)$	Cofunctions of $(\pi/2 - \alpha)$	Functions of $(\alpha \pm \pi)$	Functions of $(\pi - \alpha)$
$\pm\sin\alpha = \cos(\alpha \mp \pi/2)$	$\sin\alpha = \cos(\pi/2-\alpha)$	$-\sin\alpha = \sin(\alpha \pm \pi)$	$\sin\alpha = \sin(\pi-\alpha)$
$\pm\cos\alpha = \sin(\alpha \pm \pi/2)$	$\cos\alpha = \sin(\pi/2-\alpha)$	$-\cos\alpha = \cos(\alpha \pm \pi)$	$-\cos\alpha = \cos(\pi-\alpha)$
$-\tan\alpha = \cot(\alpha \pm \pi/2)$	$\tan\alpha = \cot(\pi/2-\alpha)$	$\tan\alpha = \tan(\alpha \pm \pi)$	$-\tan\alpha = \tan(\pi-\alpha)$
$-\cot\alpha = \tan(\alpha \pm \pi/2)$	$\cot\alpha = \tan(\pi/2-\alpha)$	$\cot\alpha = \cot(\alpha \pm \pi)$	$-\cot\alpha = \cot(\pi-\alpha)$
$\pm\sec\alpha = \csc(\alpha \pm \pi/2)$	$\sec\alpha = \csc(\pi/2-\alpha)$	$-\sec\alpha = \sec(\alpha \pm \pi)$	$-\sec\alpha = \sec(\pi-\alpha)$
$\pm\csc\alpha = \sec(\alpha \mp \pi/2)$	$\csc\alpha = \sec(\pi/2-\alpha)$	$-\csc\alpha = \csc(\alpha \pm \pi)$	$\csc\alpha = \csc(\pi-\alpha)$

Functions of $[\alpha \pm (n)(\pi/2)]$ (n** is even)	Functions of $[\alpha \pm (n)(\pi/2)]$ (n** is odd)	Reciprocal Identities	Pythagorean Identities
* $\sin\alpha = \sin[\alpha \pm (n)(\pi/2)]$	* $\sin\alpha = \cos[\alpha \pm (n)(\pi/2)]$	$\sin\alpha \csc\alpha = 1$	$\sin^2\alpha + \cos^2\alpha = 1$
* $\cos\alpha = \cos[\alpha \pm (n)(\pi/2)]$	* $\cos\alpha = \sin[\alpha \pm (n)(\pi/2)]$	$\cos\alpha \sec\alpha = 1$	$\sec^2\alpha - \tan^2\alpha = 1$
* $\tan\alpha = \tan[\alpha \pm (n)(\pi/2)]$	* $\tan\alpha = \cot[\alpha \pm (n)(\pi/2)]$	$\tan\alpha \cot\alpha = 1$	$\csc^2\alpha - \cot^2\alpha = 1$
* $\cot\alpha = \cot[\alpha \pm (n)(\pi/2)]$	* $\cot\alpha = \tan[\alpha \pm (n)(\pi/2)]$	** "n" is any integer. * Algebraic sign is determined by quadrant in which the angle falls.	
* $\sec\alpha = \sec[\alpha \pm (n)(\pi/2)]$	* $\sec\alpha = \csc[\alpha \pm (n)(\pi/2)]$		
* $\csc\alpha = \csc[\alpha \pm (n)(\pi/2)]$	* $\csc\alpha = \sec[\alpha \pm (n)(\pi/2)]$		

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$e^{j\alpha} = \cos\alpha + j\sin\alpha$

Addition Formulas

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \tan\alpha \pm \tan\beta / 1 \mp \tan\alpha\tan\beta$

$\cot(\alpha \pm \beta) = \cot\alpha\cot\beta \mp 1/\cot\beta \pm \cot\alpha$

Product Formulas

$\sin\alpha\cos\beta = 1/2\,[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\cos\alpha\sin\beta = 1/2\,[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

$\cos\alpha\cos\beta = 1/2\,[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

$\sin\alpha\sin\beta = 1/2\,[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$\tan\alpha\tan\beta = (\tan\alpha + \tan\beta)/(\cot\alpha + \cot\beta)$

$\cot\alpha\cot\beta = (\cot\alpha + \cot\beta)/(\tan\alpha + \tan\beta)$

Sum and Difference Formulas

$\sin\alpha \pm \sin\beta = 2 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} 1/2(\alpha+\beta) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} 1/2(\alpha-\beta)$

$\cos\alpha \pm \cos\beta = \pm 2 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} 1/2(\alpha+\beta) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} 1/2(\alpha-\beta)$

$\tan\alpha \pm \tan\beta = \sin(\alpha \pm \beta)/\cos\alpha\cos\beta$

$\cot\alpha \pm \cot\beta = \sin(\beta \pm \alpha)/\sin\alpha\sin\beta$

Double Angle Identities

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{2\cot\alpha}{1+\cot^2\alpha}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \tan\alpha\,(1+\sec 2\alpha) = \frac{2\cot\alpha}{\cot^2\alpha - 1}$$

$$\cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha} = \frac{\cot\alpha - \tan\alpha}{2}$$

$$\sec 2\alpha = \frac{\sec^2\alpha}{2-\sec^2\alpha} = \frac{\csc^2\alpha}{\csc^2\alpha - 2} = \frac{1+\tan^2\alpha}{1-\tan^2\alpha} = \frac{1}{2\cos^2\alpha - 1}$$

$$\csc 2\alpha = \frac{\tan\alpha + \cot\alpha}{2}$$

Half - Angle Identities

A minus sign must be prefixed to the radical if the trigonometric function of $\alpha/2$ to be found is negative

$\sin\alpha/2 = \sqrt{(1-\cos\alpha)/2}$	if $\sin\alpha/2$ is +
$\cos\alpha/2 = \sqrt{(1+\cos\alpha)/2}$	if $\cos\alpha/2$ is +
$\tan\alpha/2 = \sqrt{(1-\cos\alpha)/(1+\cos\alpha)} = (1-\cos\alpha)/\sin\alpha = \sin\alpha/(1+\cos\alpha) = \csc\alpha - \cot\alpha$	if $\tan\alpha/2$ is +
$\cot\alpha/2 = \sqrt{(1+\cos\alpha)/(1-\cos\alpha)} = (1+\cos\alpha)/\sin\alpha = \sin\alpha\,(1-\cos\alpha) = 1/(\csc\alpha - \cot\alpha)$	if $\cot\alpha/2$ is +
$\sec\alpha/2 = \sqrt{2/(1+\cos\alpha)}$	if $\sec\alpha/2$ is +
$\csc\alpha/2 = \sqrt{2/(1-\cos\alpha)}$	if $\csc\alpha/2$ is +

Square Identities

The notation $\sin^2\alpha$ means $(\sin\alpha)^2$

$\sin^2\alpha = (1-\cos 2\alpha)/2 = (1+\cos\alpha)(1-\cos\alpha) = \sec^2\alpha/(\sec^2\alpha + \csc^2\alpha) = 1/\csc^2\alpha$

$\cos^2\alpha = (1+\cos 2\alpha)/2 = \sin^2\alpha + \cos 2\alpha = \cos\alpha\sec\alpha/(1+\tan^2\alpha) = \cot^2\alpha/(1+\cot^2\alpha) = (\csc^2\alpha - 1)/\csc^2\alpha$

$\tan^2\alpha = (1-\cos 2\alpha)/(1+\cos 2\alpha) = \sec^2\alpha - 1 = (\sec 2\alpha - 1)/\sec 2\alpha + 1)$

$\cot^2\alpha = (1+\cos 2\alpha)/(1-\cos 2\alpha) = (1+\cot^2\alpha)/(1+\tan^2\alpha) = (\csc\alpha - \sin\alpha)/\sin\alpha$

$\sec^2\alpha = 2/(1+\cos 2\alpha)$

$\csc^2\alpha = 2/(1-\cos 2\alpha)$

Power Series

(α is a number of radians)

$\sin\alpha = \alpha - (\alpha^3/3!) + (\alpha^5/5!) - \cdots + (-1)^{n+1}[\alpha^{2n-1}/(2n-1)!] + \cdots$	
$\cos\alpha = 1 - (\alpha^2/2!) + (\alpha^4/4!) - \cdots + (-1)^n[\alpha^{2n-2}/(2n-2)!] + \cdots$	
$\tan\alpha = \alpha + (\alpha^3/3) + (2\alpha^5/15) + (17\alpha^7/315) + \cdots$	$-\pi/2 < \alpha < \pi/2$
$\cot\alpha = (1/\alpha) - (\alpha/3) - (\alpha^3/45) - \cdots$	$-\pi < \alpha < 0$ or $0 < \alpha < \pi$
$\sec\alpha = 1 + (\alpha^2/2) + (5\alpha^4/24) + (61\alpha^6/720) + \cdots$	$-\pi/2 < \alpha < \pi/2$
$\csc\alpha = (1/\alpha) + (\alpha/6) + (7\alpha^3/360) + \cdots$	$-\pi < \alpha < 0$ or $0 < \alpha < \pi$