# Chapter 3: Slide Rule ABC's and D's 

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Slide rules consist of sets of logarithmic scales that are used to add logarithms of numbers in order to perform multiplication and division, as well as other quick calculations. As described in the previous chapters, adding distances on the rule can be equated to adding logarithms of numbers. Now that logarithms are understood and the values of common logarithms can be readily computed, creating a logarithmic scale is straightforward. For instance, if one wanted to make a logarithmic scale that was 10 inches in length, where " 1 " was on the left end and " 10 " was on the right end, then a mark for the number $x$ would be placed at a distance of (10 inches $) \times \log _{10}(x)$ from the left-hand side: the " 2 " mark would be placed ( 10 inches $) \times \log _{10}(2)=3.01$ inches from the left end, the " 4.5 " mark would be placed ( 10 inches $) \times \log _{10}(4.5)=6.532$ inches from the left end, and so forth for any number one wants to mark on the rule. The last mark would be at ( 10 inches $) \times \log _{10}(10)$ $=10$ inches as desired.

## Making a Log Scale



To get a feel for how to read the scales, some example numbers are indicated in the above figure by the red lines at values of $1.13,1.65,3.8,7.1$ and 9.6 . As was discussed in [Review of the Logarithm], if we created a scale like the one in the above figure but went between values of 10 to 100 , or perhaps between values of 0.1 to 1 , the spacing of the marks on the scale would be the same. Hence, the red lines could in principle represent values of $11.3,16.5,38,71$ and 96 , or $0.113,0.165,0.38,0.71$ and 0.96 as well, depending upon the interpretation of the values of 1 and 10 at the left and right ends, respectively.

Edmund Gunter of England is credited as having made the first straight logarithmic scale of this kind in 1620, with which he used "dividers" (compasses) to measure and add up logarithms to perform calculations. In 1624, William Oughtred, also of England, arranged two Gunter rules and slid them back and forth (holding them together by hand) to do the same calculations without having to "measure" anything with dividers. Over the next several years devices were created to hold these sliding rules together, and circular variations of the basic design were created as well. It took over a century before cursors were added as an aid for lining up and reading results more accurately.
Without going through all the variants of the slide rule over these first two centuries following its invention, we'll simply go directly to the standard Mannheim-style rule, introduced by Amédée Mannheim of France in about 1850. The Mannheim rule has four basic logarithmic scales, most often labeled A, B, C, and D. An example of such a rule is shown below.


Figure 1: Mannheim-style K\&E Slide Rule, Model 4058W.

Notice that each scale has a layout similar to that created above. This is our standard "logarithmic scale". The A and B scales are identical to each other, and the C and D scales are again identical to each other. The $\mathrm{C} / \mathrm{D}$ scales cover one complete factor of 10 , while the $\mathrm{A} / \mathrm{B}$ scales have the same logarithmic scale pattern but is repeated and hence cover a factor of 100 . This will be important later. On the particular rule shown and on many others, the $\mathrm{A} / \mathrm{B}$ scales are labeled from 1 to 1 (interpreted as from 1 to 10 ), then 1 to 1 again (interpreted as from 10 to 100); on some slide rules " $1,2, \ldots 10$ " and " $20,30, \ldots 100$ " are written out explicitly.

The Mannheim scale layout and its labeling became standards that lasted over the remaining lifetime of the slide rule industry. These are the basic scales used for calculations involving multiplication, division, and for finding squares and square roots. The other scales we will discuss below are on many slide rules, but these four are found on essentially every slide rule.

## Basic Multiplication and Division

To multiply two numbers we can use one set of scales, either A and B, or C and D. Let's try A and B first. Suppose we want to multiply two numbers, each of which are between 1 and 10 : Let's say $2.3 \times 6.8$. To do so,

- slide the 1 on the B scale to line up with 2.3 on the A scale
- slide the cursor to line up with 6.8 on the B scale
- follow the cursor line to view the number under the cursor on the A scale; this is the final answer: 15.6 (to three digits)


Figure 2: A Multiplication Example.

If we zoom out for a moment, we can see what is going on. To find the product, we are actually adding the logarithms of the two numbers in question, and finding out which number has a logarithm equal to that sum.


Figure 3: Addition of logarithmic distances.

Note that it may be hard to tell on the A scale whether the answer is closer to 15.5 or 15.6 , etc., due to the granularity of the scale one is reading, especially for numbers on the right-hand end of the scale. So, one may have to "guess" at that third decimal place.

And that's one reason for the $\mathrm{C} / \mathrm{D}$ scales. Since one decade (factor of 10) on these logarithmic scales is twice as long as those on the $\mathrm{A} / \mathrm{B}$ scales, the $\mathrm{C} / \mathrm{D}$ scales have more accuracy and hence might really be important for getting that third digit correct. In general, the scale or scales on the slide rule that have the greatest length for a single factor of 10 are called the $\mathrm{C} / \mathrm{D}$ scales. Some rules, called long-scale rules, can have C/D scales that are 50 cm ( 20 inches) rather than 25 cm ( 10 inches), and some can have scales wound up in circular, spiral or helical forms which can give overall scale lengths much longer for higher accuracy.

But 25 cm is generally considered the "standard" length and gives a typical accuracy of about three digits. ${ }^{1}$ Even smaller lengths (which can fit in a pocket, for instance) can be found, but while convenient they are seldom as accurate. (See [Non-Linear and Long-Scale Rules] for examples of long-scale rules, and [Pocket Rules] for examples of smaller-scale rules.)
So, let's do the same multiplication calculation using the $\mathrm{C} / \mathrm{D}$ scales:

- slide the 1 on the C scale to line up with 2.3 on the D scale
- slide the cursor to line up with 6.8 on the D scale. Oops! What just happened?
- Because the C/D scales only go from 1 to 10 , the above operation takes us off scale!
- No problem; since every decade of a logarithmic scale has the same spacing of divisions, we simply need to imagine starting the problem from the left and then keep track of our factors of 10 . So, let's repeat the operation, but this time...
- slide the $\mathbf{1 0}$ on the C scale to line up with 2.3 on the D scale instead of using the $\mathbf{1}$
- slide the cursor (to the left!) to line up with the 6.8 on the C scale
- now, follow the cursor line to view the number under the cursor on the D scale; this is the answer (within a factor of ten).

You'll see that the answer that you read off will be 1.56 on the D scale. But because we lined up with the right-hand-edge of the scale rather than the left, there is an extra factor of ten involved. Also, since you were multiplying 2.3 and 6.8 you know that the answer should be something closer to 15 , so the correct answer must be 15.6.


Figure 4: Using the C/D Scales.

Two things to learn here: The C/D scales give more accuracy in most cases than the A/B scales. So, they'd be the "go-to" scales. Also, it is always important to take a guess at what the answer should be so that one

[^0]gets the right power of ten in the end. The slide rule will keep track of the digits in the answer; the user will have to keep track of the powers of 10 .

General Rule: To multiply, find the value of the first number on the D scale, then line up a " 1 " on the C scale with this first number, using whichever end of the C scale gives access to the second number (also on the C scale). Slide the cursor to the second number as found on the C scale. Follow the cursor line to read off the answer on the D scale directly below.

What about division? Let's take 6.8 divided by 2.3 as our example. We follow these steps (using C/D):

- slide the cursor to 6.8 on the D scale
- slide the 2.3 on the C scale to line up with 6.8 on the D scale
- slide the cursor to 1 on the C scale
- follow the cursor line to view the number under the cursor on the D scale; this is the answer, again, within a factor of ten. In our case here, it is the answer, as $6.8 / 2.3$ should be about 2.96 .


Figure 5: Division using the C/D Scales.

What just happened? We found 6.8 on the rule, noting that its distance from 1 on the rule is equal to $\log 6.8$. We then subtracted a distance that corresponds to $\log 2.3$. The result is given by a length corresponding to $\log 2.96$.

General Rule: To divide, line up the cursor with the numerator on the D scale; slide the C scale so that the denominator also lines up with the cursor; the answer will be on the D scale directly below the " 1 " on the C scale.

If you think about the above operations for a while, you should realize that you are either adding logarithms to perform multiplications, or subtracting logarithms to perform divisions. Once you realize that that is what you're doing, then you'll be able to perform sequences of multiplications and divisions with lightening speed.

As an illustration, let's estimate the average speed in miles per hour of the earth in its orbit (radius $=93.0$ million miles) about the sun using a slide rule. Set up the problem by gathering factors of ten as follows:

$$
v=\frac{2 \pi R}{T}=\frac{2 \pi\left(93.0 \times 10^{6} \mathrm{mi}\right)}{1 \text { year } \times 365 \text { day } / \text { year } \times 24 \mathrm{hr} / \text { day }}=\frac{2 \times \pi \times 9.30 \times 10^{7} \mathrm{mi}}{3.65 \times 2.4 \times 10^{3} \mathrm{hr}}=\frac{2 \times \pi \times 9.30}{2.4 \times 3.65} \times 10^{4} \mathrm{mi} / \mathrm{hr}
$$

It is good practice and often useful to order the numbers in the numerator and denominator such that factors of near-equal value are above/below each other. First, this practice gives you an early impression of what the
result will be. From our numbers above, we see that multiplying and dividing these numbers should give us an answer somewhat less than 9. Also, by performing a multiplication followed by a division and then repeating using the numbers from left to right, this tends to keep the result fairly centered on the rule. So, here are the next steps using the slide rule:

- Move the cursor to 2 on the D scale + which, technically, is multiplying 1 by 2 .
- Set the C scale to 2.4 at the cursor
+ thus dividing by 2.4 , according to our rules; the answer thus far would be found on the D scale under the " 1 " on the C scale; but we don't really need that answer at the moment.
- Move the cursor to $\pi$ (3.142) on the C scale + thus multiplying the previous result by $\pi$.
- Set the C scale to 3.65 at the cursor + thus dividing the previous result by 3.65 .
- Move the cursor to 9.3 on the C scale + thus multiplying the previous result by 9.30 .

The result is read on the D scale under the cursor; it should be about 6.67 . This is consistent with our estimate that it be less than 9.

The FINAL answer is this number times $10^{4}$, or $66,700 \mathrm{mi} / \mathrm{hr}$.
Compare with result from a computer calculation:
\#\# [1] $2 *$ pi*93.0e6/24/365 $=66705.0494940299$

## Squares and Square Roots

We mentioned that the $\mathrm{A} / \mathrm{B}$ scales can be used for multiplication/division, as can the $\mathrm{C} / \mathrm{D}$ scales. But the primary use of this complete set of scales is to quickly find squares and square roots. The length of a decade (from 1 to 10 on the scale, for instance) on the A scale is half the length of a decade on the D scale. Hence, as the logarithm of a number on D advances, the logarithm of a number on A advances at twice that rate. In other words, "log $\mathrm{A} "=2 \times$ " $\log \mathrm{D}$ " which tells us that $\mathrm{A}=\mathrm{D}^{2}$.

The A scale gives the squares of the numbers found on the D scale. This is very convenient when one wants to calculate areas of squares or circles, calculate kinetic energies, calculate power when given an electrical current and resistance, etc. In reverse, for a given number on the A scale its "square root" will be found directly on the D scale. The same relationship of course holds between the B and C scales as for the A and D scales. The figure below shows a close-up of the B and C scales of our example slide rule. It is easy to spot on the $B$ scale the squares of the numbers on the $C$ scale. Is it easy to read off the value of $\sqrt{2}$ ?


Figure 6: Squares and Roots.

Suppose we wish to calculate the kinetic energy $W$, in Joules, of an object of mass $m=2.7 \mathrm{~kg}$ moving at a speed $v=6.8 \mathrm{~m} / \mathrm{s}$. The formula for kinetic energy is $W=\frac{1}{2} m v^{2}$. Let's do the calculation two different ways. First, we can use the C/D scales to multiply all the factors together:

- Line up the cursor to 5 on the D scale.
- Align 10 on the D scale with the cursor, and move the cursor to 2.7 on the C scale.
- Align the 1 on the D scale with the cursor, and move the cursor to 6.8 on the C scale.
- Align the 10 on the D scale with the cursor, and move the cursor to 6.8 on C again.
- Find the result at the cursor on the D scale: 6.25 .
- Estimate the answer: a bit less than $3 / 2 \times 49 \approx 75$.
- So, final answer is 62.5 Joules.

The above calculation took three settings of the slide. Alternatively, we can calculate the result with a single setting of the slide (in this particular case) utilizing the A scale. Start by placing the cursor on 2.7 on the A scale. Then, move the slide to line up the 2 on the B scale with the cursor. The result of dividing 2.7 by 2 will be found on the A scale, aligned with the 1 on the B scale. But note that the square root of this result is also found directly below on the D scale. Since the 1 on the C scale is also aligned with this result, we can now multiply by 6.8 by moving the cursor to 6.8 on the C scale. At this point we have on the D scale - under the cursor - the result of $\sqrt{m / 2} \times v$. The square of this number will be directly above on the A scale. It should read roughly $\frac{1}{2} m v^{2}=62.5$ (Joules) and is our final answer.
Yet again, via computer:
\#\# [1] $1 / 2 * 2.7 * 6.8^{\wedge} 2=62.424 \mathrm{~J}$

Notice, too, that the kinetic energy for other speeds as well, from $1 \mathrm{~m} / \mathrm{s}$ up to about $8.6 \mathrm{~m} / \mathrm{s}$, can be read without having to move the slide.

Another common calculation is that of the length of the hypotenuse $c$ of a right triangle given the lengths of the other two sides, $a$ and $b$, where $c^{2}=a^{2}+b^{2}$. Suppose two sides have lengths $a=3.7$ and $b=5.2$ and we want to calculate $c$. One way is to use the A scale and find the square of $a$ (13.7) and the square of $b$ (27), and - writing down the results if necessary - add the two results together (40.7). Then, move the cursor to this result on the A scale and find its square root directly below on the D scale: about 6.35 .

Another approach is to rewrite the relationship as

$$
c=\sqrt{a^{2}+b^{2}}=a \sqrt{1+(b / a)^{2}}
$$

where we note that $b>a$ for the computation that follows. This time, using just two settings of the slide and without having to write down any intermediate results, we can find the hypotenuse. Here are the steps to perform on the slide rule:

- Move the cursor to 5.2 on the D scale.
- Align 3.7 on the C scale to the cursor, thus dividing 5.2 by $3.7(=b / a)$.
- Sliding the cursor to the 1 on the C scale, we follow the cursor to the A scale to read the square of this result: $1.98\left(=(b / a)^{2}\right)$.
- Mentally adding 1 to the result, we move the cursor to 2.98 on the A scale, $1+(b / a)^{2}$.
- The square root of the present result will be directly below on the D scale, $\sqrt{1+(b / a)^{2}}$; so next we move the slide to align the 1 on the C scale with the cursor.
- Moving the cursor to 3.7 on the C scale we are multiplying the previous result by 3.7 (a).
- The final answer, $c=a \sqrt{1+(b / a)^{2}}$, will be at the cursor on the D scale: $c=6.38$.

Check:
\#\# [1] $\operatorname{sqrt}\left(3.7^{\wedge} 2+5.2^{\wedge} 2\right)=6.38200595424354$

As a matter of fact, the same technique for the hypotenuse calculation can be made with only one setting of the slide. Take one number, say the larger one (b), and find it on D. Line up the middle index on B (the " 10 ") with this number. Then, using the cursor, find the second number (a) on D. Read off the number under the cursor that is now found on B; call this $n$. Then, slide the cursor over to the number " $n+10$ " on the B scale. Under the cursor on D will be $\sqrt{a^{2}+b^{2}}$. Let's apply this to our specific example:

- Move the cursor to 5.2 on the D scale.
- Align the middle index on B to the cursor.
- Now find 3.7 on the D scale using the cursor.
- Read the number $n$ under the cursor on B. It should be $n=5.05$.
- Move the cursor to $n+10=15.05$ on the B scale.
- remember to interpret the middle index as "10" on B, and then find 15.05
- Under the cursor, on D, read the answer: $\sqrt{5.2^{2}+3.7^{2}}=6.38$.


Figure 7: Setting to perform the above calculation.

This last example is actually an old technique, best arrived at by the law of proportions. (See discussions found in the vignettes entitled [Out of Proportions] and [Carpenters vs. Engineers], particularly, [Comments on Scale Development].) It was more commonly taught in Europe, while our preceding method more likely would have been taught in the U.S. schools during the mid-20th century. So how does this work? The numbers on the $\mathrm{A} / \mathrm{B}$ scales are squares of the numbers on the $\mathrm{C} / \mathrm{D}$ scales, and hence if we line up a number $N_{B}$ on B with a number $N_{c}$ on C this corresponds to a ratio of $\sqrt{N_{B}} / N_{C}$. With the slide set in this position, all other numbers on B and C are set to the same ratio.

So, for instance, in our figure above, the slide is set to this common set of proportions:

$$
\frac{\sqrt{n}}{a}=\frac{\sqrt{10}}{b}=\frac{\sqrt{n+10}}{x}
$$

and we would like to solve for $x$ by eliminating $n$ from these equations. Take the first and second ratios; they give $n=10(a / b)^{2}$. Hence, the system can be re-written as

$$
\frac{\sqrt{n}}{a}=\frac{\sqrt{10}}{b}=\frac{\sqrt{10} \sqrt{(a / b)^{2}+1}}{x} .
$$

Taking the last two ratios and dividing by $\sqrt{10}$ gives us

$$
x=b \sqrt{(a / b)^{2}+1}=\sqrt{a^{2}+b^{2}}
$$

and there we have it!

## Gauge Marks

Many slide rules have additional markings on their scales called "Gauge Marks" (sometimes called gauge points). A gauge mark indicates a commonly used numerical value that can be found quickly for use in certain computations. The most common mark is usually for the value of $\pi=3.14159 \ldots$. On the rule used in the examples above, we see a $\pi$ symbol on the A and B scales. Consider its use for the following (which is similar in technique to our last example): Set the 1 on the B scale to the $\pi$ gauge mark on the A scale. On the D scale directly below the $\pi$ will be $\sqrt{\pi} \approx 1.77$. Now look at numbers along the C scale; take 4 , for instance. The number below it on the D scale will be $\sqrt{\pi} \times 4 \approx 7.09$. Looking directly up on the A scale we see the "square" of this result, which will be $\pi \times 4^{2}=7.09^{2} \approx 50.3$. That is, with the B scale gauged to $\pi$ on the A scale, the A scale will give the area of a circle for any radius read off from the C scale. Check: Suppose the radius is 2 ; then without moving the slide at all one can find the value of 2 on the C scale and directly read off the area $\pi \times 2^{2}=12.5$ on the A scale right above. A great time saver when repeating many similar calculations!


Figure 8: Using a $\pi$ Gauge Mark.

Other gauge marks might be for $\sqrt{\pi}=1.772$ and $\sqrt{\pi / 4}=0.886$, which are often used for calculating cross sections and volumes, etc.; $e=2.718$, for exponentials; Watts/Horsepower $=746$ (or, 7.46 on the rule), for power conversion calculations; degrees/radian $=57.30$ (or, 5.73 on the rule), for trigonometry calculations, and many others.

Some slide rules have very few if any gauge marks, while some slide rules have many. And while there was a bit of consistency in the labeling of certain special numbers ( $\pi$ was most always marked as $\pi$ ), there certainly was no set standard and so many variations are seen. In fact some slide rules might have a "mark" on the rule, but the mark does not have a label at all. The definitive guide to the values and labels of Gauge Marks on slide rules can be found in the Pocketbook of the Gauge Marks, by Panagiotis Venetsianos, published by the Oughtred Society and available on its web site.

## Cubes and Cube Roots - the K Scale

The Mannheim slide rule became very popular after its standardization in the late 1800 's and soon rule makers began adding other scales to ease various computations. One that is often found on later rules is the K scale, used for cubes and cube roots. The K scale is similar to the A scale in that it is just a logarithmic scale that repeats three times over the length of the standard C scale, rather than just twice. By the same type of argument as made above for the A scale, the K scale shows the cube of a number found on the D scale; conversely, lining up the cursor on a number on the K scale, one can find that number's cube root directly on the D scale. The image below, showing a $\mathrm{K} \& E$ slide rule with $\mathrm{K}, \mathrm{A} / \mathrm{B}$, and $\mathrm{C} / \mathrm{D}$ scales, can be used to illustrate the point.


Figure 9: $\mathbf{A} \mathbf{K \& E}$ slide rule with $\mathbf{K}, \mathbf{A} / \mathbf{B}$, and $\mathbf{C} / \mathbf{D}$ scales (plus an inverted scale, CI).

For practice, let's repeat our calculation of the volume of a sphere of radius 4.58 inches, as was performed during our review of logarithms, but here using a slide rule. Using the $\mathrm{C} / \mathrm{D}$ scales, we can compute the value of $4 \pi / 3$, which should be about 4.189 . Some slide rules might have the cube root of this value (1.612) engraved as a Gauge Mark on the K scale. The user next moves the 1 on the C scale to line up with this value on the K scale using the cursor. Then the cursor is moved along the C scale to the value of the radius of the sphere, which is 4.58 in our case. The volume can be read directly on the K scale, which should be roughly 400 cubic inches. Using the K scale the result may only be accurate to two digits for our example.

## The Log Scale - L

Many slide rules have the L scale on them, which gives the logarithm of the number aligned typically on the C or D scale. As can be seen from our plot labeled Logarithmic Scale in [Logarithms and Log Scales], if the values on the C scale have a logarithmic spacing, then the logarithms of those values will have a linear spacing. Hence, the $L$ scale on a slide rule is easy to identify as it has its digits 0 to 10 spaced evenly by about one inch on a standard 10 inch rule.

An early use of the $L$ scale was to find the value of a number raised to an arbitrary power. Suppose you wanted to know $3.7^{4.6}$. Then one could find the $\log$ of $3.7=0.568$ on the L scale and multiply by 4.6 to get 2.614 using the C and D scales. Then find on the L scale which number has the logarithm 0.614 - which is 4.11 - and then multiply by 100 (due to the " 2 " out front in 2.614 ) to get the final answer: 411 . Check by computer:
\#\# [1] 3.7 to the power $4.6=410.89223180408$.

We noted in previous sections that sometimes reading squares and cubes from the A and K scales can yield results only accurate to $1-2$ digits. If a more accurate answer is required, the L scale often can be used to perform the calculation. For example, let's re-compute the volume of a sphere of radius $R=4.58$ inches. Rather than using the K scale, which yielded "about 400 " cubic inches, we do the following:

- From our formula for the volume of a sphere, $V=\frac{4 \pi}{3} R^{3}$, we see that $\log V=\log (4 \pi / 3)+3 \log R$.
- If a gauge mark is not already on the rule, find our necessary constant of $4 \pi / 3$ using the $\mathrm{C} / \mathrm{D}$ scales. It should be about 4.19.
- Using the L scale, find the $\log$ of 4.58 , which is about 0.661 , and the $\log$ of 4.19 , which is about 0.622 .
- Using C/D, or mentally, find 3 times 0.661 , which should be roughly 1.983 .
- Adding these results, $0.622+1.983=2.605=0.605+2$.
- Resetting the slide if necessary, use the L scale to determine the number whose $\log$ is 0.605 . I find about 4.03; the " 2 " in the above result tells us to use 403. A more precise answer, using a personal computer, is 402.425 .

We see that using the $L$ scale for this purpose may give more accuracy, but it also involves many more steps and perhaps the writing down of intermediate values. If one only needs a more approximate answer, the use of the $A / B$ and $K$ scales can save significant time.

## Inverse and Folded Scales

One often finds inverse scales on a slide rule, such as CI and DI. The CI scale, for instance, is just the C scale printed in reverse. Since the logarithm of the inverse of a number is equal to the negative of the logarithm of the original number:

$$
\log x^{-1}=-1 \times \log x=-\log x
$$

then the CI scale can be used to find the reciprocal of any number on the C scale. This can also be used in multiplication/division problems since dividing by a number is equal to multiplying by its reciprocal. That is, by using the procedures above for multiplying numbers with the rule but by using the appropriate inverse scale, one is actually dividing. This can often help speed up multi-step calculations.
Another aid to multi-step calculations can be the use of folded scales, such as CF and DF. As we saw earlier, sometimes adding or subtracting logarithms leads to an answer that is off the end of the rule. So, rather than moving the slide to the left or right to realign during a complex calculation, one might opt to use the appropriate CF or DF scale; these scales have the identical numerical spacings as the C or D scales and so perform the same function, but they begin and end in the middle of the normal $\mathrm{C} / \mathrm{D}$ scale range.

The geometric middle of the C or D scale (which goes from 1 to 10 ) will be equal to the square root of 10 $=3.162$. Many slide rule manufacturers realized early on that it might be more convenient to use $\pi$ as the fold-point rather than $\sqrt{10}$, thus creating an easy way to multiply by $\pi$ which is quite often useful, and this became common on the more modern slide rules.

Let's do a couple of examples:

1. Multiply 2.5 times 6.2 times 9.4. Let's start by aligning the index on C with 9.4 on D , and moving the cursor to 2.5 on C. To perform our next multiplication we could reset the slide by aligning the left index on C to the cursor. However, our next factor - 6.2 - would be off-scale on C. If, instead, we put the index of CF at the cursor, then the 6.2 is on-scale on CF. Moving the cursor to 6.2 in CF thus gives us our final answer under the cursor on D: 146.
2. Multiply 1.5 times 3.7 times 6.4. Using the D and CI scales, align 3.7 on CI with 1.5 on D . This is similar to our "division" technique, but here we are dividing by the inverse of 3.7 and so it is actually a multiplication. The answer, still, would be found on the D scale below the index on CI. But, the index on CI is aligned with the index on C. Thus, without even moving the slide, we can multiply by our third number by moving the cursor to 6.4 on the C scale and finding the final answer below on the D scale: 35.5.

One might also find combinations of the scales above on some rules, such as CIF or DIF. The Aristo slide rule pictured below contains CI and CIF scales as well as DF/CF scales, for example. This gives the user many options when performing a long series of consecutive multiplications and divisions.


Figure 10: An Aristo slide rule that includes CI, CIF, DF, and CF scales. Note, also, the L scale (log) along the bottom.

The sequence of numbers on these inverse and folded scales are just the same as those on the standard C and D scales, but start at a different "phase" or end-point, or they are read in an opposite direction. Some manufacturers called slide rules with the CI scale a polyphase or maniphase rule (particularly by K\&E and Dietzgen, respectively). Note that the A/B and K scales should be called "harmonics" of the base scale. It's the CI scale, the CF/DF scales, and other similar scales and combinations that are truly the same scale with a different phase and hence it is these that create a so-called multi-phase slide rule.

## Trigonometric Scales

Other than L, the scales discussed thus far have been simple logarithmic scales. They are used for finding products, certain powers, or certain roots of numbers. To make the slide rule more complete - particularly for the type of problems encountered in navigation, science, and engineering - scales for finding values of basic trigonometric functions were incorporated into the rules. And once again, by utilizing the logarithms of these functions, direct calculations involving trigonometric functions could be performed.

The figure below defines the basic trigonometric relationships for an angle $\theta$ between two line segments: the sine $(\sin \theta)$, cosine $(\cos \theta)$, and tangent $(\tan \theta)$ of the angle.


Values of these trigonometric functions for angles between zero and 90 degrees are plotted here:

## $\sin \theta$ (black) $\cos \theta$ (blue) $\tan \theta($ red $)$



The vertical dashed lines are at $\theta=45$ and 90 degrees.
Scales for use in determining the sine and tangent of an angle are often found on basic slide rules. It is hard to describe all the variations of such scales in a single paragraph or subsection, as variations do exist. Suffice it to say that most scientific slide rules have a sine scale and a tangent scale of some kind and one usually can tell from their markings how to use them if one is familiar with basic trigonometry. Let's see how such a set of scales could be implemented on a slide rule.

The values of the sine function range from 0 to 1 for angles between 0 and 90 degrees, and the values of the tangent function range from 0 to 1 for angles between 0 and 45 degrees. Consider then the C or D scale which can be interpreted to range from 0.1 to 1 . By creating a new scale, usually labeled S (sine) that goes from about 5.7 degrees (because $\sin 5.7392^{\circ}=0.1$ ) to 90 degrees, one can set the cursor to a certain number of degrees $x$ on the S scale and read $\sin x$ on the D scale, say. A separate scale - the T scale - can similarly be used to read the tangents of angles. Note that $\tan 5.7106^{\circ}=0.1$.

By taking the above curves for sine and tangent, exchanging the $x$ and $y$ axes, and plotting the $x$-axis with a logarithmic scaling, we get the following:

Sine (blue), Tangent (red)


In the plot above, the circles and crosses are drawn at 10-degree intervals for the sine and 5 -degree intervals for the tangent, respectively. The horizontal axis now corresponds to the D scale on the rule, and the degrees that correspond to its values are shown on the graph. These values make up the S and the T scales. As we can see, both the S and the T scales will begin at roughly $5.7^{\circ}$ and end at $90^{\circ}(\mathrm{S})$ or $45^{\circ}(\mathrm{T})$ as expected. If we take the circles and crosses and lay them out on a ruler with the spacing given above, we have created the S and T scales found on most slide rules:


Another way of thinking about these scales is to ask the following: For a certain value of $x$ on the D scale, at what angle $\theta$ does the sine (or tangent) have this value? In other words, we want $x=\sin \theta$, or $\theta=\sin ^{-1} x$. Markings for the numbers on the D (or C ) scale are placed a distance $L_{0} \log x$ away from the left index, where $L_{0}$ is the total length of the logarithmic scale (typically 25 cm , or about 10 inches). So, numbers on the $S$ scale will be placed a distance $L_{0} \log (\sin \theta)$ from the same index, and the $T$ scale's markings will be $L_{0} \log (\tan \theta)$ accordingly. From the earliest years of the development of the logarithm, tables of the logarithms of trigonometric functions were included in the books of logarithms because of the usefulness of these values in navigational and astronomical calculations. It is not surprising that the S and T scales were incorporated into the early Mannheim-style slide rules in addition to the standard logarithmic scales.

For angles less than about $5.7^{\circ}$, the sine and the tangent of such small angles are virtually identical. If the angles are expressed in radians, then $\sin x \approx \tan x \approx x$. So a third scale - called ST - is sometimes found on slide rules for use with small angles. To three significant digits, $\sin 0.573^{\circ}=\tan 0.573^{\circ}=0.010$. So, the ST scale has angles from about half a degree up to about 5.7 degrees, and the numbers on the D scale will need to be interpreted as between 0.01 and 0.1 . The graph below shows how close the two values of sine and tangent are for small angles.

## The ST scale: blue =S red = T



The earliest slide rules had angles marked off in degrees, minutes ( $1 / 60$ of a degree) and seconds (1/60 of a minute). More modern rules used decimal degrees rather than the antiquated minutes and seconds of arc. Additionally, in the mid-1900s, the use of radians rather than degrees to describe angles became more common with many engineers and scientists. In a complete circle there will be $360^{\circ}$ or $2 \pi$ radians. With decimal degrees on the ST scale, it is straightforward to convert degrees to radians and vice versa, by reading degrees on the ST scale and radians on the D scale, since for small angles $\sin x \approx x$. To emphasize this feature many slide rule makers relabeled the ST scale as SRT, with the " R " standing for radians. Additionally, some rules can have Gauge Marks to allow for quick conversion between degrees and radians, while certain rules might even have other specific scales for conversion between these quantities or for direct readings using radians.

On some of the older slide rules, and on many "beginner's" rules, the appropriate scales of degrees were placed on the back of the slide. On some models, by lining up the "degree" number on a line within a small window on the back of the rule the sine and tangent of that angle appears on the front of the rule, usually read off of the C scale at the end of the A or D scale. Some of the older models as well as some of the "beginner's" rules also expect the values to be read from the A scale; when this is the case, then the entire range of $\sin x$ from 0.01 to 1 is on the A scale, and thus interpreting a third digit is often hard.

Now if the S and T scales are on the back, the slide could be removed from the stock and re-inserted with the S and T scales facing the front of the rule, making it easier to read directly or to use directly in a calculation. Such juggling became unnecessary with the invention of "duplex" slide rules, which had a double-sided cursor and coordinated hairlines so that the scales on one side were aligned with the scales on the other side. It was common for the S and T scales to be on the slide; hence, rather than just "reading off" the values of the trig functions, one can immediately multiply or divide these values in calculations.
Up to now we have been talking about sines and tangents of angles. To find the cosine of an angle $x$ between 0 and $90^{\circ}$ ( 0 to $\pi / 2$ radians), $\cos x=\sin \left(90^{\circ}-x\right)=\sin (\pi / 2-x)$. Also, $\cos x=\sin x / \tan x-$ an operation that can be performed rather straightforwardly. As an example, suppose we wanted the cosine of 38 degrees. If that's all we need, we can look up the sine of $90-38=52$ degrees, which is 0.788 ; this is the cosine of 38
degrees. Another approach would be to use the slide rule as follows:

- With the S and D scales lined up, set the cursor on the S scale to 38 degrees; the sine of this angle can be read on the D scale (if you want to know it).
- Now move the slide (not the cursor) to line up 38 degrees on the T scale at the cursor hairline.
- The cosine of 38 degrees can now be read on the D scale, directly under the 1 on the C scale.

Why does this work? Logarithms! The C/D scales are logarithmic, and so when we perform the second step above, we are dividing the tangent into the sine of the angle - and this is $\sin \theta / \tan \theta=\cos \theta$ !
So long as the trig scales are located on the slide of the slide rule, such tricks can be used to perform many common operations. For example, suppose we want to find the hypotenuse $c$ of a right triangle given the sides $a$ and $b$. For such a calculation, suppose $b<a$. Then, define $\theta$ such that $\tan \theta=b / a$. The hypotenuse will be $c=b / \sin \theta$. Let's take something we know the answer to: $a=4$, and $b=3$. Traditionally, one might set the cursor to 4 and read, on the A scale, its square - 16 - and write that down. Then, do the same with the other number, 3 to get its square -9 . Then, add them together on paper to get 25 . Next, find 25 on the A scale and read its square root -5 on the D scale below. There are other methods for finding the hypotenuse, but here we want to do the calculation using the rule's trig scales which are assumed to be on the slider of the rule:

- Set the cursor to the smaller number, 3 on the D scale.
- Divide by the other number, 4 , by moving 4 on the C scale to line up with the cursor. Move the cursor to the end of the C scale to line up with the result (0.75).
- Reset the slide in order to look up the angle $\theta$ on the T scale which has this tangent - you should see about 36.9 degrees.
- Now, move the S scale so that this same angle lines up with the original smaller number, 3 , using the cursor.
- Looking on the D scale just below the " 1 " on the C scale, you should find 3 divided by the sine of our $\theta$, the result of which should be - say it with me -5 !

Some of the advanced slide rules may have a special scale (usually called the P scale, or Pythagorean scale) that gives the value of $\sqrt{1-x^{2}}$ for a given value of $x$ on the D scale. This is a common operation for certain other computational problems, but noting that $|\cos x|=\sqrt{1-\sin ^{2} x}$ means that one can use this scale to quickly find the cosine of an angle. An example of a rule with a $P$ scale is provided here:


Figure 11: Castell with a "P" Scale.

The cursor in the figure above is aligned with the 40 degree mark on the S scale. We see that the sine of this angle is 0.643 (on the D scale). On the P scale we can read off the cosine of this angle as 0.766 .
Lastly, it is also important to remember that the slide rule does not take into account the "sign" of its arguments, and so as is typical of many trigonometric problems one must understand the procedures for operating in the various quadrants where angles may be greater than 90 degrees or have negative values, etc.

## Log-Log Scales

In the section entitled The Log Scale - L we discussed how the L scale can be used to find numbers raised to arbitrary powers. This involved looking up a log, multiplying the result by a number, then looking up what number has that log. Special scales called "log log" scales were created to make this a simpler operation. Suppose we seek a number $z$ which is given by a number $x$ taken to the power of $y$ :

$$
z=x^{y} .
$$

If we take the logarithm of each side,

$$
\log z=\log x^{y}=y \times \log x,
$$

and by taking the logarithm again we find that

$$
\log \log z=\log y+\log \log x
$$

Since we already have scales on the slide rule proportional to $\log y$ (the C scale, for instance), then by adding a scale that is proportional to the $\log \log$ of a number (that is, the logarithm of the logarithm of $x$ ) we can perform the exponential operation essentially in one step.
An example of a log-log scale is shown here (on the DeciLon slide rule, by K\&E):


Figure 12: Log-Log scales at the bottom of the DeciLon rule.

Suppose we want to compute $3.5^{2.5}$. Find the $\log -\log$ scale (called Ln on this rule, it can also be designated by LL, or N, or perhaps other labels) at the bottom which has 3.5 in it. Note that these scales do not automatically repeat like standard "log" scales. So, you have to find the scale with the actual number of interest on it. On the rule we are using we find the number " 3.5 " on the $\operatorname{Ln} 3$ scale at the very bottom. Now, by placing the 1 on the C scale directly above 3.5 on the Ln3 scale, and sliding the cursor over to 2.5 on the C scale, we can read off " 22.9 " on the Ln3 scale directly. As you can readily imagine, we have added the log of 2.5 to the $\log -\log$ of 3.5 to get the $\log -\log$ of 22.9 . Done!


Figure 13: Exponentiation calculation using the Log-Log scale.

Well, we were actually a bit lucky there. First of all, perhaps 23 is as close a guess as one can read on this scale. The log-log scales aren't always easy to read accurately, especially on the right-hand side. But also, one will often find that the answer appears to be off scale. Just as in multiplication and division problems, the ends of the slide can be exchanged when the result is off scale, after which the result can be read on the next log-log scale in the series. For each new log-log scale with larger values along it, the C/D scales represent an additional factor of 10 in the exponent. For example, had we wanted to take $3.5^{0.25}$, we would find the answer on the Ln2 scale, directly above our previous answer: approximately 1.368 .
As another example, consider a calculation of the number $1.065^{35}$. With the cursor at 1.065 on the Ln1 (or LL1) scale, we must move the cursor to the left to reach 3.5; the value of $1.065^{3.5}$ will now be found one scale higher, on the Ln2 (or LL2) scale: about 1.246. The particular result we are seeking, $1.065^{35}$, is found on the Ln3 (or LL3) scale: approximately 9.06.
Of the 330 slide rules presently in the Collection, approximately $38 \%$ of them have log-log scales. More details about the creation and use of $\log -\log$ scales can be found in the vignette [Power Rules]. Also, several very good synopses of the use of $\log -\log$ scales to find numbers raised to powers can be found online; one Log-Log example page is based upon Pickett slide rules and covers the basics.

But a very important point that needs to be made in our present discussion is that the bases used in the log-log scales can be mixed. For instance, if $z=x^{y}$ and one takes the natural $\operatorname{logarithm}\left(\right.$ Base $\left.e: \quad \ln x \equiv \log _{e} x\right)$, then

$$
\ln z=y \times \ln x
$$

since the general rules of logarithms still apply. But then, taking the common (Base 10) log of the result, we find

$$
\log \ln z=\log y+\log \ln x
$$

Since the C scale utilizes Base 10, then the mix of logarithm bases still allows powers of numbers to be found using the exact same technique described above - this is because the number $x$ and the number $z$ are both read from the same scale or set of scales.

But now let's let $x=e$ in our equation above. Then

$$
\log \ln z=\log y+\log \ln e=\log y+0=\log y
$$

In this case, $\ln z=y$, or $z=e^{y}$. So with these mixed bases, if a number $y$ is read on the D scale, then $e^{y}$ will be on the log-log scale. Hence such mixed-base log-log scales can be used to take numbers to arbitrary powers as well as to read values of the exponential function for a wide range of arguments. Additionally, by using the scales in reverse, finding the power that $e$ must be raised to in order to arrive at a particular number is equivalent to finding the natural logarithm of that number. The log-log scales can be used for all three of these tasks.

Look back at the figure above to view the Ln3 scale again. At the left end one can find the " $e$ " Gauge Mark. It is located directly below the value of " 1 " on the D scale. This shows us that the Ln scales are mixed-base on this rule, since if the D scale has value $x$, with $1 \leq x \leq 10$, then this $\operatorname{Ln} 3$ scale will correspond to $e^{x}$. The $\operatorname{Ln} 2$ scale will give $e^{x / 10}, \operatorname{Ln} 1$ will be $e^{x / 100}$, and $\operatorname{Ln} 0$ will be $e^{x / 1000}$.
The earliest slide rules with log-log scales usually only had LL1 (or LL0) through LL3. If one needed numbers like, say, $e^{-3}$, then one would find $e^{3}$, record or remember that value, and then use the CI scale to find the inverse. Eventually scales of $e^{-x}$ were added, typically labeled LL03, LL02, LL01, and LL00 for direct reading of such values. They can also be used for the evaluation of hyperbolic trig functions.

## Hyperbolic Trigonometric Scales

On a select number of slide rules, scales of values of the hyperbolic trigonometric functions might be found. These scales, typically labeled SH and TH (or Sh and Th), operate much like the normal trigonometric scales S and T . The physical length from the index to a value $x$ on the SH scale, for instance, is proportional to the logarithm of the value of the function $\sinh (x)$. The hyperbolic trig functions are defined in terms of the exponential function:

$$
\sinh x=\frac{e^{x}-e^{-x}}{2} \quad \cosh x=\frac{e^{x}+e^{-x}}{2} \tanh x=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}
$$

Again, like the basic trig functions, the cosh function is typically not found on the slide rule; it can be computed from $\cosh x=\sinh x / \tanh x$.


Figure 14: K\&E Model 4083-3 Vector Rule (back).

The hyperbolic trig functions appear in computations such as in the case of a rope attached at two ends and hanging under its own gravity (called a catenary), or in various computations that entail exponential growth or decay. For instance, from the above definitions it can be seen that

$$
e^{x}=\cosh x+\sinh x, \quad \text { and } \quad e^{-x}=\cosh x-\sinh x
$$

In addition, sometimes problems require the calculation of trigonometric functions of complex numbers. A complex number has a real and an imaginary component. The result of such an operation results in a new complex number with its own real and imaginary components, and these components will depend upon hyperbolic functions of the original components. It is actually this particular application for which slide rules with the hyperbolic functions were first created, primarily for use in electrical engineering. As this is a very special use of the hyperbolic functions, further discussion of these so-called "vector" calculations is reserved for the vignette [Vector/Hyperbolic Calculations].

## Special Rules and Scales

The above sections describe most of the primary scales that can be found on typical slide rules. A basic slide rule might have a limited number of scales, such as those found on the Mannheim layout. Some of the more advanced slide rules might have 10-20 scales on them, with folded scales and log-log scales and perhaps hyperbolic trig scales. However, similar to how a computer program or application might be used today, many other specialty slide rules also were developed for particular types of repetitive calculations found in various professions.

## Specialty Rules

An example of a specialty rule would be an "Electro" slide rule, which is a type that was made by various manufacturers for the specific use in the electrical trades. It has Gauge Marks on certain scales corresponding to the resistivity of copper, assuming diameters of wire in inches or millimeters, for instance, and lengths in feet or meters. Knowing the standard voltage used in the system, voltage drops, currents, and power levels can be computed quickly.
Other examples include "chemical" slide rules with built-in marks indicating commonly used densities and other properties of various compounds for use in chemistry; "radio" and "electronics" rules used in the radio and electrical engineering industries with special scales for relationships between frequencies, capacitance, decibels, and other common variables; stadia rules used in surveying, which provided direct calculations of horizontal offset and vertical rise from transit readings; "artillery" rules used in the military; "proportions" rules used in typography and engraving applications, and so forth.

In many cases the [Specialty Slide Rules] can be quite rare and are often highly sought after by collectors. (Personally, I'd love to have a Black Body radiation slide rule!)


Figure 15: Hemmi Electro.


Figure 16: Hemmi Chemical Rule.


Figure 17: Engraving Rule.


Figure 18: Pickett Model N-515-T Electronics Rule.

## Long Scales

To complete our tour of the subject, there are slide rules with basic scale arrangements but with very long scales for reading values to more significant figures. Some of these are themselves quite long - 20 inch Mannheim-type rules were common in the early part of the 20th century. But circular slide rules also can give higher accuracy with scales that can be up to a factor of $\pi$ longer than their overall dimension. Spiral scales (such as on the Boykin RotaRule and even helical scales (such as the Otis King's Calculator and the Fuller Calculator) can give total scale lengths of up to several feet, providing multiplications and division with impressive accuracy. Examples can be found in the section [Non-Linear and Long-Scale Rules]. See also the vignette, [The Long and Short of It].

## Segmented Scales

Some of the more modern rules have certain scales that are broken up into segments, so that a 20 -inch scale can fit onto a 10 -inch rule, for instance. One of my favorite slide rules is the Pickett Model 4. (See figure below.) It has the following scales on it:

## Front:

- a 30-inch equivalent cube/cube-root scale
- a 20 -inch equivalent square/square-root scale
- 20-inch equivalent T and $\mathrm{S} / \mathrm{ST}$ scales
- standard 10-inch C/D, CI/DI, CF/DF scales


## Back:

- standard 10 -inch C/D, CI scales
- $\mathrm{CFm} / \mathrm{DFm}$ scales - folded at value of $\ln 10$ for conversion between Base $e$ and Base 10
- 80 -inch equivalent log-log scale
- TH and SH scales for direct reading of hyperbolic trigonometric functions and direct vector calculations And all in Eye-Saver yellow with black text!


Figure 19: Pickett Model 4-ES (front).


Figure 20: Pickett Model 4-ES (back).

## More Information

Slide Rule manufacturers of the 20th century usually provided instruction booklets/leaflets for the use of their slide rules, many of which are hard to come by 100 years later. Often times expanded texts - in either hard-bound or soft-bound varieties - were written and sold by the manufacturers as well, and often can be found for sale in used book stores. And of course there were several authors who wrote books on the subject without the bias of a single manufacturer. The Chapter [Books, Manuals, and Sheets] contains a few books which I have stumbled upon over the past few years and have added to the collection.

The International Slide Rule Museum (ISRM) has produced dozens of books with reproductions of original slide rule manuals and of original catalogs from major slide rule manufacturers as part of their Slide Rule Instructions Library Preservation Project. A searchable Table of Contents of this collection is available at the ISRM web site.


The reader is also encouraged to check out the other slide rule information found in the [References] section of this web site. The "Beginner's Guide to Collecting Slide Rules" and "All About Slide Rules", both produced by the Oughtred Society, are particularly useful introductory publications.

Over the past three chapters, we have gone through the mathematical formalism of the logarithm and its computation, and we have seen how the logarithm allows for the generation of scales that can be used to perform multiplication and division on a slide rule. Armed with these new insights, and with additional slide rule books and information within reach, hopefully the collection of slide rules displayed in the following two chapters can be viewed with more meaning and appreciation.


[^0]:    ${ }^{1}$ The Englishman William Oughtred invented the slide rule, and the standard became a 10-inch rule - 10 inches for every factor of 10 . Eventually, the more conventional standard became metric -25 cm per factor of 10 .

