Chapter 2: Computing Logarithms

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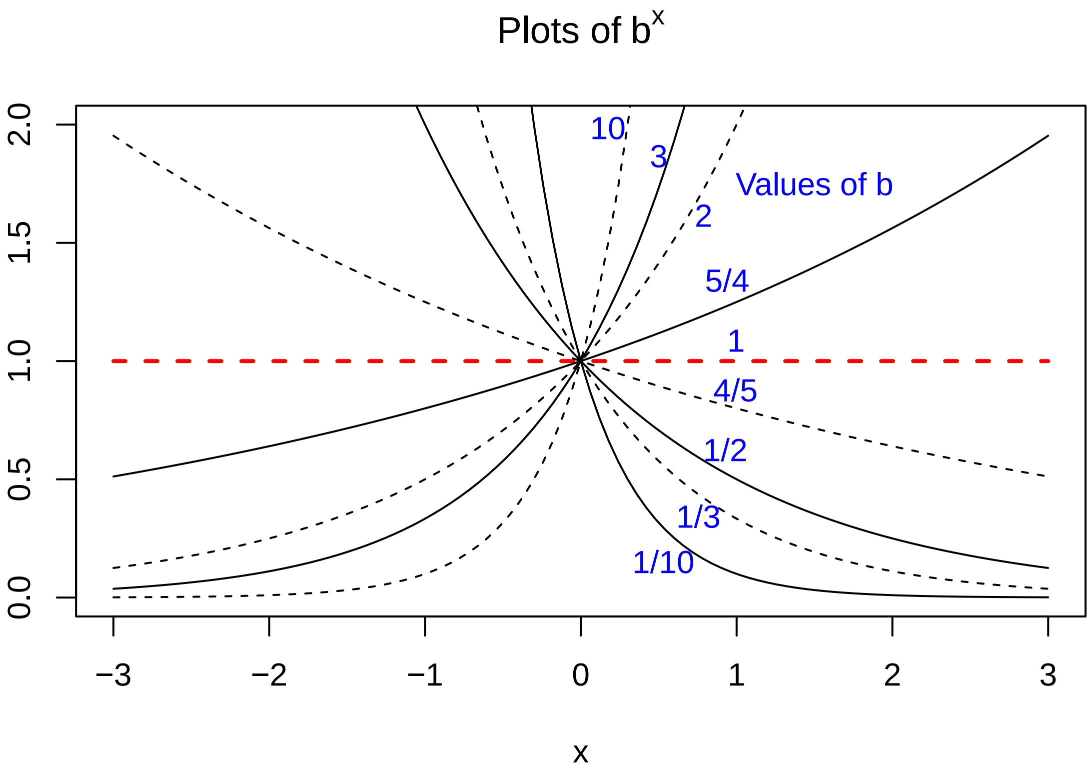
In the last chapter we gave examples of how one could fill in a curve of the form $x = b^p$ in a laborious iterative fashion and, by *inverting* the result, one could in principle obtain the exponent p (the logarithm) that corresponded to a given x for that particular base. Rather than taking such an iterative approach and "filling in" tables of numbers[1], we would rather be able to generate a formula for a logarithm of any given number (and, in fact, for any chosen base) and be able to compute it to any desired accuracy.

In what follows we will use *calculus* to find a *natural* base to use for our computations. With our appropriate definition of a natural logarithm we can use a standard technique to find a *Taylor Series* in terms of the argument x to create a formula for computing the natural logarithm of x using our natural base. Then, by using one of our general rules of logarithms, we can find the *common* (Base 10) logarithm of the number x to any reasonably desired accuracy. Those readers not versed in the mathematics of the *calculus* should not be deterred from reading through the following sections. Equations for computing logarithms, including example lines of computer code, are presented which might still be of interest to some readers. So even though the details presented might appear esoteric to the non-expert, they are provided for completeness of the topic of the logarithm.

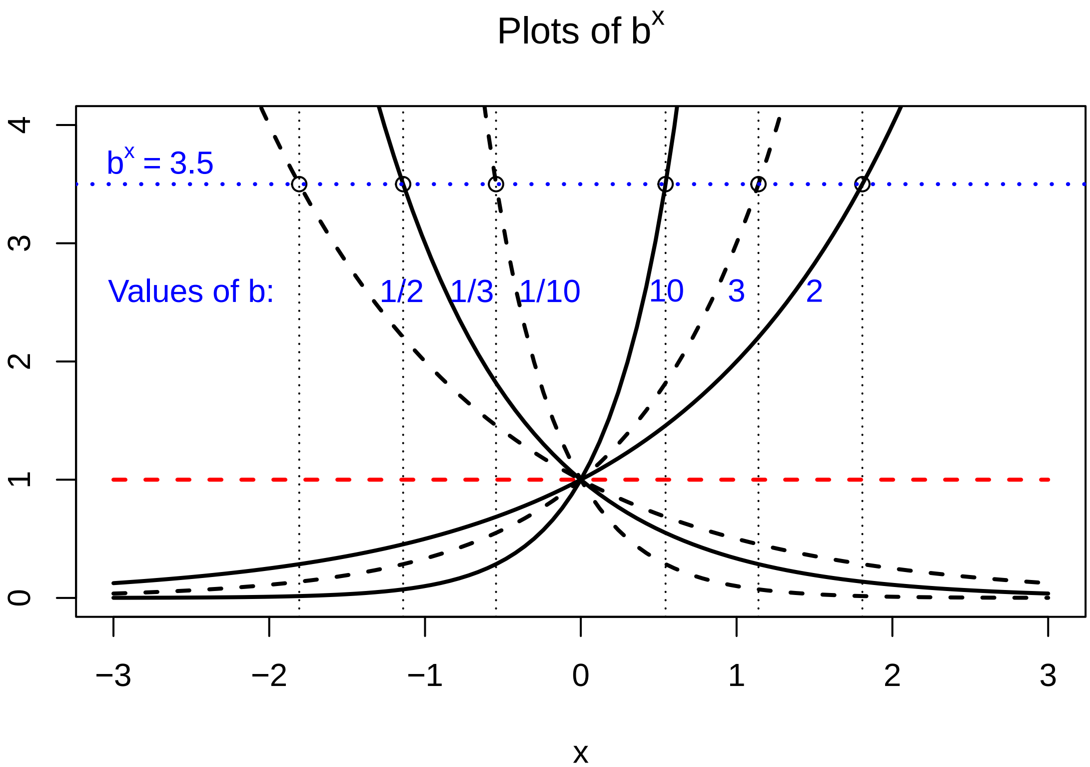
[1] It is estimated that Napier spent 20 years developing his first complete table of logarithms! And the connection between Napier's approach and an "exponential" approach would not be made for yet another 20 years after that.

A Word About Bases

If we make plots of b^x for various values of b we can see that the curves all look very similar in nature:



The above plot emphasizes that for any base b, $b^0 = 1$ and $b^{-x} = (1/b)^x$. Let's draw a horizontal line across a similar plot, say at a value of 3.5, and look at where it intersects a few of our curves:



The above figure illustrates how the number 3.5 can be represented by any base raised to a particular power. As particular examples,

- $3.5 = (0.5)^{-1.8074} = (0.333)^{-1.1403} = (0.1)^{-0.5441}$,

and

- $3.5 = (10)^{0.5441} = (3)^{1.1403} = (2)^{1.8074}$.

Obviously, if we were to choose a different value for our base b, then we could see where its curve crosses the line at 3.5 and hence know the logarithm of 3.5 (the exponent, x) for that particular base.

To see how the various choices of base are related, take the operation b^x and suppose we wish to write this in terms of a new base, a. Taking

$$\log_a b^x = x \ \log_a b$$

and raising a to this power yields

$$b^x = a^{x\ (\log_a b)}.$$

We see that b^x is the same as $a^{x'}$ with a special scaling of the argument, namely $x' = x \times \log_a b$. But is there a natural choice for a particular base number, and its logarithms, from which all other exponential functions should be scaled?

Our quest below will be to find a *natural* definition for a logarithm and to determine its *natural* base, then show how to compute values of logarithms for that base. Once this is done, we can find the logarithm in any other base through the general rules that any logarithm must obey. This will also give us the tool needed to directly compute the value of any number taken to any arbitrary power.

The Natural Logarithm

We want to find a mathematical function $f(x)$ that can be evaluated in a straightforward way, and that has the properties of a logarithm that were discussed in the previous chapter. For instance, the function would need to have the following properties:

- The logarithm of 1 must zero for any choice of base. Hence, $f(1) = 0$.

- The function should be positive for values of x greater than one, and negative for values less than one: $f(x < 1) < 0; \quad f(x > 1) > 0$.

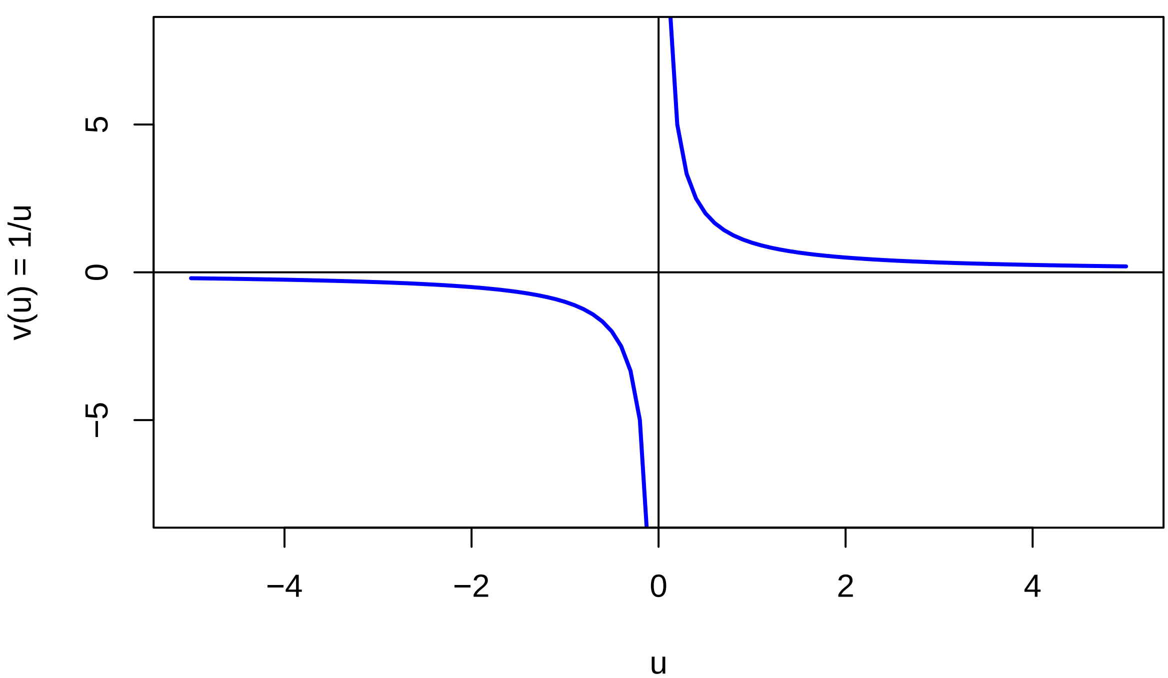
- As x approaches zero from the positive side, the function should approach negatively infinite numbers: $f(x) \to -\infty$ as $x \to 0$.

- And, as x gets more and more positive, the function should tend toward infinite numbers: $f(x) \to +\infty$ as $x \to +\infty$.

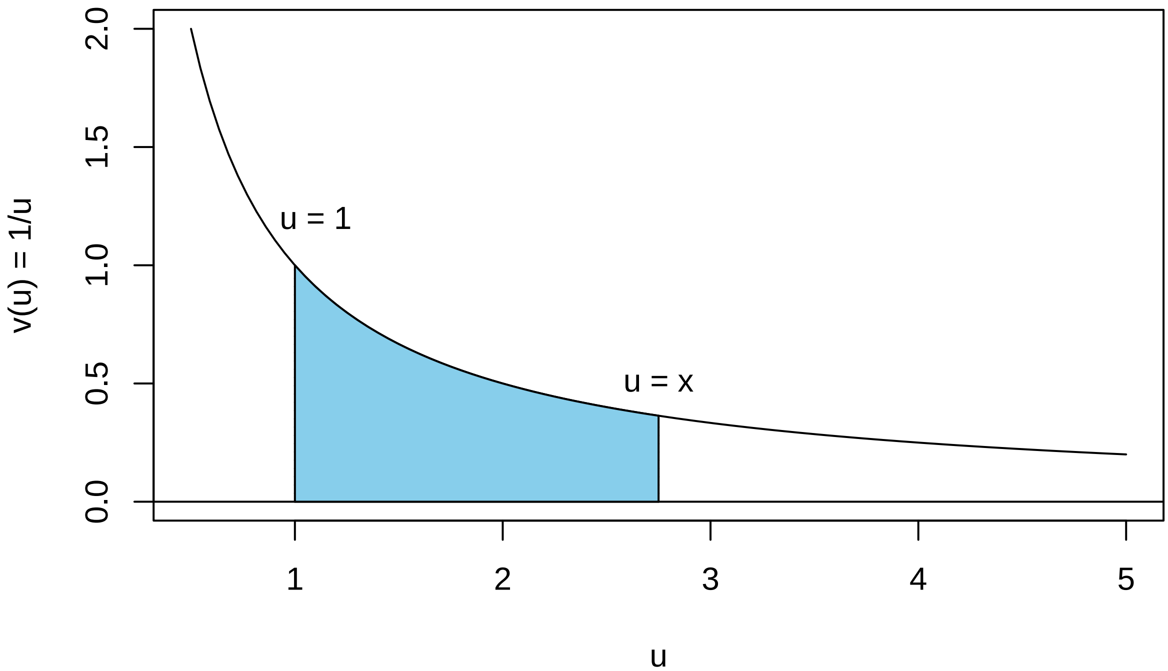
- Lastly, but perhaps most importantly, since the logarithm of a product of two numbers must be the sum of the logarithms of the two numbers, we must have: $f(a \times b) = f(a) + f(b)$.

It turns out that the area under a specific geometrical figure, which can be accurately computed through the use of *calculus*, has the above properties and hence can be used to define a natural logarithm. This may seem like a very strange approach toward defining a logarithm which is, after all, an exponent applied to a base number, a seemingly unrelated mathematical exercise. But we'll see that for one particular curve, these two operations have a remarkable connection and the power of *calculus* allows us to compute the answers in a straightforward way.

The curve in question is that of a hyperbola. In terms of the coordinates in a plane, u and v, the equation of a hyperbola can be written as $u \cdot v = 1$ or, as a function of u, $v(u) = 1/u$.



If we take the branch of the hyperbola for positive values of u we find, after a bit of inspection, that our properties of a logarithm can be met by defining a new function that is evaluated by finding the *area* between the hyperbolic curve and the horizontal axis. The area in question is shown below:



The shaded area always starts at $u = 1$ and stops at an arbitrary value x, and so yields a function of x, call it $f(x)$. This function, therefore, can be computed using *calculus* by performing the following integration:

$$Area = f(x) = \int_1^x \frac{1}{u}\,du.$$

This integral has the properties of a logarithm as listed above. First of all, it obeys the rule of a logarithm in that the logarithm of $a \times b$ is equal to the sum of the individual logarithms of a and of b:

$$\begin{aligned} f(a \times b) &= \int_1^{ab} \frac{1}{u}\,du \\ &= \int_1^{a} \frac{1}{u}\,du + \int_a^{ab} \frac{1}{u}\,du = \int_1^{a} \frac{1}{u}\,du + \int_1^{b} \frac{1}{w}\,dw \qquad (w \equiv u/a) \\ &= f(a) + f(b). \end{aligned}$$

Secondly, for $x = 1$ the integration range will have zero length and hence $f(1)$ will be zero. For values of $x < 1$ the integral will be negative, just as logarithms are negative for $x < 1$. In addition, the integral goes to $-\infty$ as x approaches 0 and to $+\infty$ as x approaches $+\infty$. The function $f(x)$ given by the above integral obeys all the basic properties of a logarithm.

For this reason, this integral represents a *natural logarithm* and is given the designation $\ln x$ with the formal definition,

$$\ln x \equiv \int_1^x \frac{1}{u}\,du.$$

The Base of the Natural Logarithm

As discussed in the previous chapter, a logarithm is an exponent of a base number that gives a desired value after the exponentiation is performed. So before proceeding to find actual values of the natural logarithm, an immediate question arises. If $\ln x$ as defined above is a logarithm, what is the *base* of this logarithm? We give this special base number the symbol "e" recognizing that it will be used to describe the natural exponential operation. For any real number $x > 0$ the number e and the natural logarithm $\ln x$ should satisfy

$$\ln x = \log_e x = p, \qquad \text{if} \qquad x = e^p.$$

This tells us that $\ln e = 1$, another property of any logarithm and its base. So geometrically we can see that, equivalently, if we perform our integration above between the limits of 1 to e, we should arrive at a value of 1:

$$\ln e = \int_1^e \frac{1}{u}\, du \equiv 1.$$

So we could, in principle, vary the value of the upper limit of the integration until a value of 1 is achieved for the integral, to an acceptable accuracy. This indeed can be done using iterative numerical methods and would provide a value for the base number e. However, let's take a slightly different approach. Let's consider that $\ln x$ and e^x are inverse functions. That is, given a value x, if we take e^x and then take its natural logarithm, we get x back again. So, in general,

$$\ln e^x = x.$$

We now perform another *calculus* operation and take the *derivative* with respect to x on both sides of the above equation. Since our logarithm is defined as an integral of the function $1/x$, then the derivative of $\ln x$ will be $1/x$. This in itself is an interesting result:

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

But go back to our previous relationship, $\ln e^x = x$, and let's take the derivative of both sides:

$$\frac{d}{dx} \ln e^x = \frac{d}{dx} x.$$

Through the chain rule for derivatives, the *left-hand side* of this equation is simplified by defining the function $v = e^x$ and then

$$\frac{d}{dx}\ \ln v = \frac{1}{v}\frac{dv}{dx} = \frac{1}{e^x}\ \frac{d}{dx} e^x.$$

But since the derivative $\frac{d}{dx} x$ is just equal to 1, then re-equating the two sides we get

$$\frac{1}{e^x}\ \frac{d}{dx} e^x = 1$$

or,

$$\frac{d}{dx} e^x = e^x.$$

The derivative of the function e^x **is** e^x !

This remarkable result gives us all we need to easily compute e (and e^x) to any desired accuracy. To do so, we use another result from *calculus*, namely a **Taylor Series** expansion. If the values of the derivatives of a function $f(x)$ are known at a point $x = a$, then it can be shown that

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)\ (x-a)^2 + \frac{1}{3!}f'''(a)\ (x-a)^3 + ...$$

where $f'(x)$, $f''(x)$, *etc.*, are the first, second, *etc.*, derivatives of $f(x)$, while

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

is the *factorial* of n. For our case we take $a = 0$ and $f(x) = e^x$. Then, since each derivative will be e^x as well, we find that $f(0) = f'(0) = f''(0) = ... = e^0 = 1$. We thus arrive at a straightforward expression for the exponential function in terms of an infinite series containing terms of x raised to integer powers:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + ...$$

And, for $x = 1$, we have a way to compute the value of our natural base,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + ...$$

Let's compute e using a computer program. We'll look at the results when using a variable number of terms, `nTerms`, in the above infinite series, up to 12 terms, and look for when the series converges to within 6 decimal places:

```
eEst  = cumsum( 1/factorial( seq(0,11) ) )
eErr  = eEst - exp(1)
```

nTerms	eEst	eErr
1	1.000000	-1.7182818
2	2.000000	-0.7182818
3	2.500000	-0.2182818
4	2.666667	-0.0516152
5	2.708333	-0.0099485
6	2.716667	-0.0016152
7	2.718056	-0.0002263
8	2.718254	-0.0000279
9	2.718279	-0.0000031
10	2.718281	-0.0000003
11	2.718282	0.0000000
12	2.718282	0.0000000

We see that the series converges to within 6 decimal places after about 11 terms or so, with a final value of about $e =$ `eCalc` $= 2.718282$.

We have just computed a fairly accurate approximation of the base number for the natural logarithm.[2] As a check, if we integrate $1/x$ from 1 to e we should get 1, since $\ln e = 1$ if it is the base number. We can do that with our computer programming language which has an integration function built in:

```
fx = function(x){ 1/x }

integrate(fx, 1, eCalc)
```

```
## 1 with absolute error < 1.1e-14
```

The base number, e, for the natural logarithm is called *Euler's number*, so-named after the Swiss mathematician Leonhard Euler who implemented the use of this symbol.[3] It is also sometimes called *Napier's constant* as John Napier actually arrived at its value during his pioneering studies into logarithms about 100 years earlier. Napier's original "base" number was actually $1/e$ in his calculations of logarithms. His colleague Henry Briggs, in 1615, convinced Napier to revise his system of logarithms to use Base 10, for which $\log 10 = 1$ and thus simplifies the use of logarithms in everyday computations.[4] The properties of the number e, only a few of which we have mentioned, make it a special number indeed, and it is regarded as one of the most important numbers in all of mathematics.

[2] In actuality, e is an irrational number, as is π, and hence its decimal representation will never repeat.

[3] Euler was indeed the first to use this symbol, but it isn't clear if he meant for the "e" to stand for his name, or if it stood for "exponential", or if it was just the next vowel to use after "a", which was the other symbol he used in his work at the time. He is credited for using the symbols "f(x)" to define a function, for instance, as well as "i" for $\sqrt{-1}$, among many other standard notations.

[4] See *The Development of Mathematics*, Dover 1992, (second edition, McGraw-Hill, 1945), pp. 161-162.

Computing Natural Logarithms

A process for computing the natural logarithm of the number x is now prescribed. Using the Taylor Series technique once again, we can take derivatives of the function $y(u) = 1/(1+u)$ and express the result in a power series form. Doing so, it can be shown that for values of $-1 < u \leq 1$,

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + u^4 - ...$$

We'll use this fact to compute the integral defining $\ln x$:

$$\begin{aligned}
\ln x &= \int_1^x \frac{1}{z}\, dz = \int_0^{x-1} \frac{1}{1+u}\, du \qquad (u = z-1) \\
&= \int_0^{x-1} \left(1 - u + u^2 - u^3 + u^4 - ...\right)\, du \\
&= \left(u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \frac{1}{5}u^5 - ...\right)\Bigg|_0^{x-1} \\
&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - ...
\end{aligned}$$

And so, by replacing x with $1+x$ on both sides of the equation, we can write

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - ...$$

where, again, this formula is only valid for $-1 < x \leq 1$. For $x < 0$ all of the terms in the above power series are negative, and so as $x \to -1$ the logarithm $\ln(1+x) \to -\infty$. (In fact, the series does not converge for $x = -1$.) For $x = 0$ we get $\ln 1 = 0$ as we should. And we can compute a logarithm for any number greater than zero up to 2 (i.e., $x = 1$). Thus, natural logarithms for all relevant numbers greater than zero and up to 2 can be computed to any required accuracy using the above power series.

But what about a value for $\ln 3$, for instance? Well, $\ln 3 = \ln(2 \times 1.5) = \ln 2 + \ln 1.5$ and so $\ln 3$ can be computed by addition of these two logarithms. What about a number like $\ln 22.8$? Similarly, we can write $\ln 22.8 = \ln(2^4 \times 1.425) = 4 \ln 2 + \ln(1 + 0.425)$ and we have all that is necessary for performing this calculation. Through the use of our power series formula the natural logarithm for any number of interest may be computed to any desired accuracy, in principle.

And ultimately, what we have been interested in from the beginning is the computation of Base 10 or *common* logarithms. We know that

$$\log x = \ln x / \ln 10$$

from our basic rule for changing bases, and so if we can compute $\ln x$ then we only need to divide by $\ln 10$ to obtain the common logarithm of x. From our previous argument,

$$\ln 10 = \ln(8 \times 5/4) = \ln(2^3) + \ln 1.25 = 3 \ln(1+1) + \ln(1 + 0.25).$$

Let's do it:

```
lnX = function(x){
   x = x-1
   lnCalc = 0
   n = 0
   while(abs( x^(n+1) / (n+1) ) > 1e-7){
      n=n+1
      lnCalc = lnCalc + (-1)^(n-1) * x^n / n
   }
   lnCalc
}

ln10  = 3*lnX(2) + lnX(1.25)
```

The results:

lnX(2) = 0.69315	lnX(1.25) = 0.22314	ln10 = 2.30259

So since we have just computed the value of $\ln 10$ and since we have a general formula that can be used for computing $\ln x$, we now have everything we need to compute the *common* logarithm $\log x$ for any arbitrary positive number.

Computing Common Logarithms

For common logs, we can write any number in scientific notation and compute its logarithm by breaking it into two parts. Suppose x is written in scientific notation as $x = a \times 10^n$ where $1 \le a < 10$ and n is an integer. Then

$$x = a/10 \times 10^{n+1}$$

and

$$\ln x = \ln(a/10) + (n+1)\ \ln 10$$

from which

$$\log x = \frac{\ln x}{\ln 10} = \frac{\ln(a/10)}{\ln 10} + n + 1.$$

Since a was defined to be between 1 and 10, then $a/10$ is between 0.1 and 1 and we can use our power series expansion for $\ln(x)$ in our calculations. We are now in a position to create a function to compute common logarithms for general numbers. Here is an example computer code to do just that, re-using our previous function for computing natural logarithms:

```
logCom = function(x){
# split input into two strings:  Num = c(".827","2")
   Num = strsplit(format(x, scientific=T),"e")[[1]]
# take the 2 parts and make them numeric:  a = 0.827,
#                                          p = 2
   a   = as.numeric(Num[1])
   p   = as.integer(Num[2])
# use our formula...
   lnX(a/10)/ln10 + p + 1
}
```

Below are a few examples, using our function above, of the calculation of various common logarithms:

log 0.718 = -0.14388	log 1.25 = 0.09691	log 2 = 0.30103
log e = 0.43429	log 200 = 2.30103	log 425 = 2.62839
	log 3.4×10^5 = 5.53148	

Summary

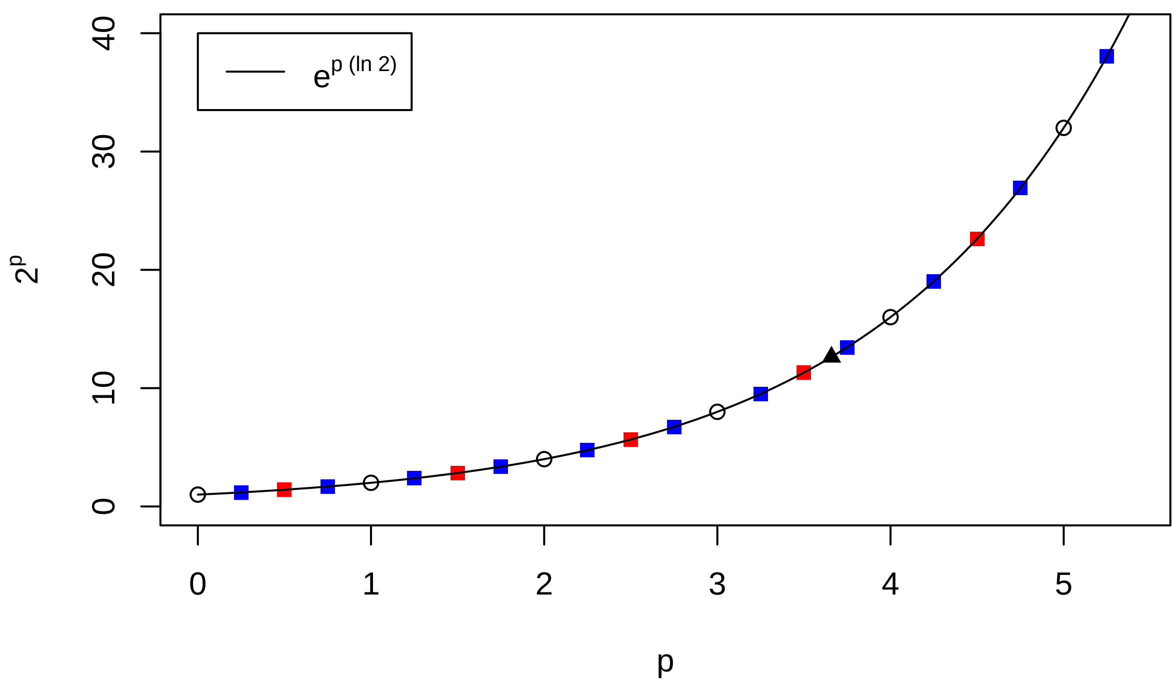
We have found that the base of the natural logarithm, e, can be computed to very high accuracy in a straightforward way, as can the values of the natural logarithm $\ln x$ of an arbitrary number x, by using a Taylor series (or, "power series") approach. Such calculations could take a very long time in the days before high performance electronic computers, yet, as we've seen, books of tables of such results were constructed centuries ago by similar means. Today, values of logarithms can be obtained directly on your smart phone.

In addition, our natural base raised to any power, e^x, can also be computed as well using the power series derived above. In fact, using another of our results, we see that any number b raised to a power x can be written in terms of the exponential function as

$$b^x = e^{x \ \ln b}$$

and so any number raised to any power can be computed using the functions developed here. Every exponential growth rate can be considered a scaled version of e^x, a function whose value grows at an instantaneous rate that is equal to its value.

Take the graph we constructed in the previous chapter where we plotted 2^p for various values of p, which included $p = 0,\ 0.25,\ 0.5,\ \ldots\ 4.75,\ 5$. To emphasize our point, we re-plot these results below, along with a curve of e^x which is a calculation based upon our Taylor series formulas, but identifying $x = p \times \ln 2$:



And finally, since we know how logarithms from different bases are related to each other, we can compute logarithms $\log x$ which use Base 10 from the logarithms $\ln x$ computed using Base e.

The point of this discussion has been to describe a process through which the common logarithm of an arbitrary number can be computed to the accuracy required for implementation on a slide rule. We developed formulas that can be used to compute logarithms $\log x$ that use Base 10, derived from logarithms $\ln x$ using Base e. We have *not* described the actual historical path of the development of logarithm calculations, and the computational algorithms shared above are not necessarily (almost guaranteed not to be) the most computationally efficient means for attaining values of logarithms. But hopefully the reader has gained an understanding for what is involved in such calculations as well as an appreciation for the work that it must have taken to arrive at values of logarithms in the days before the modern electronic computer.