Chapter 1: Review of the Logarithm

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The elementary calculations performed using slide rules utilize the concept of the logarithm, invented in the early 1600's by John Napier of Scotland.[1] Napier introduced the name *logarithm* as a combination of two ancient Greek terms *logos*, meaning "divine reason", and *arithmos*, meaning number. He was looking for a relationship between an arithmetic and a geometric progression, so that tables could be generated that would aid in performing multiplication calculations. At that time, calculations in astronomy and, perhaps most importantly, in navigation required the multiplication of numbers of up to 6-7 digits each.

To illustrate the issue suppose we want to multiply the two numbers 4873 and 382. This can be performed on paper in the following way:

			2	2		
			6	5	2	
			1	1		
			4	**8**	**7**	**3**
		×		**3**	**8**	**2**
	1	2	2			
			9	7	4	6
	3	8	9	8	4	
1	4	6	1	9		
1	**8**	**6**	**1**	**4**	**8**	**6**

Of course one needs to know the "times tables" very well. And after a few "carries" (indicated above in small type, such as in "8 times 7 is 56, carry the 5"), keeping things lined up and keeping track of what is being added requires discipline and concentration, especially for situations involving many more digits than the above. And the repeated multiplication or division of several multi-digit numbers can become very tedious very quickly. Napier's concept was to find for each number a corresponding *logarithm* such that multiplying two numbers was reduced to *adding* their logarithms. Adding two numbers, even if the numbers have many digits, is a simpler task and much less error prone.

The concept that was studied and developed by Napier can be illustrated with the following table:

p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
x	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768

The top row – the numbers designated "p" – run sequentially from 0 to 16. The bottom row – designated "x" – starts with the number 1. Then, a base number is chosen. For our example we chose "2" as our base. Next, each consecutive number in the x row is the previous result multiplied by the base number. The list created

[1] A nice discussion of Napier's development of his early tables of logarithms can be found in Denis Roegel's, *Napier's ideal construction of the logarithms* [Research Report], ffinria-00543934f (2010). See https://hal.inria.fr/inria-00543934/en .

in the second row is called a *geometric sequence.* Now notice the following. If we take two numbers in the x list and multiply them, the result falls under the *sum* of the corresponding values in the p list. For instance, take 2 times 8; the result we know is 16. The corresponding values of p in the table for the numbers 2 and 8 are 1 and 3. We see that the sum of these p values is $1 + 3 = 4$, and the number in the x row that falls below this result is $x = 16$, the result of multiplying our original two numbers. The values of p are called *logarithms* of the values of x.

Now suppose we wanted to quickly find 32 times 256. We could take out pencil and paper and write

		1	1	
		1	1	
		2	**5**	**6**
×			**3**	**2**
	1			
		5	1	2
	7	6	8	
	8	1	9	2

or, from our table above, we could look up the logarithm of 32 which is 5, and the logarithm of 256 which is 8. Adding the logarithms we find $5 + 8 = 13$. We then look at the table to find which number x has a logarithm of $p = 13$. The answer found from the table is $x = 8192$. Thus, 64 times 512 is equal to 8192.

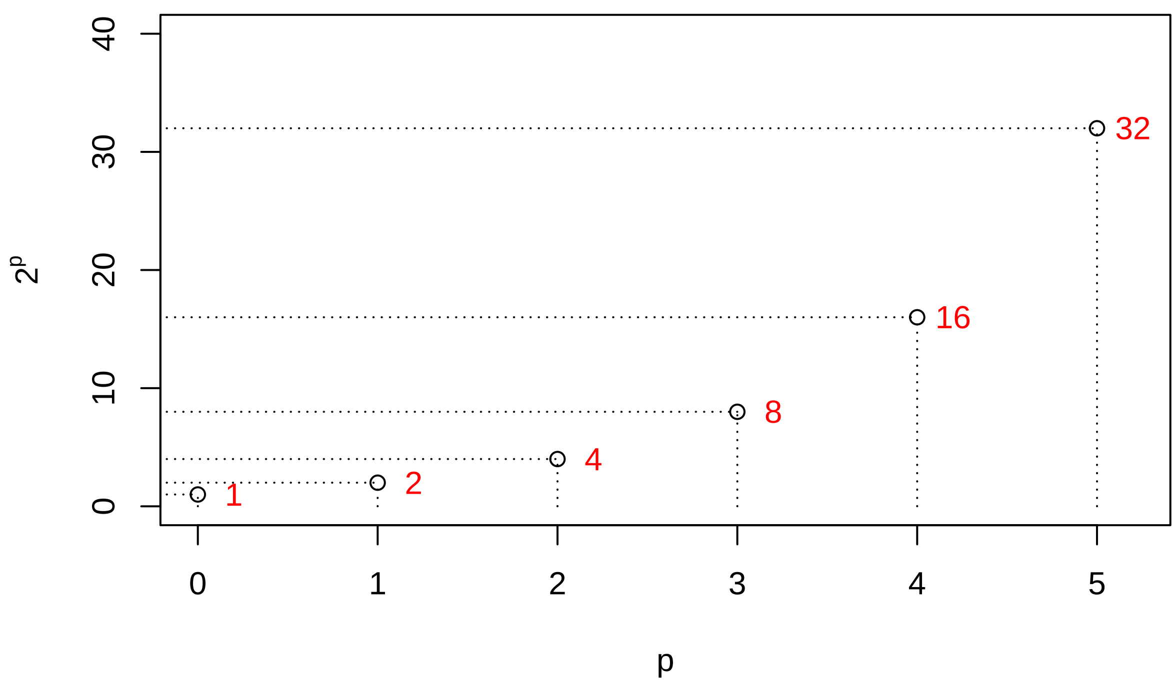
In our simple example above, the value of x that can be used from the table is a rather sparse list. The goal of Napier was to find a technique for computing logarithms of other integers in between in order to fill out a complete table for any value of x. From the table it can be surmised, for example, that using 2 as our base, the logarithm p for the number $x = 7$ is somewhere between 2 and 3. If we knew its logarithm more precisely, we might be able to perform similar computations using the number 7.

Napier's goal was to determine logarithms of the natural numbers for some suitable base number. Through many years of effort he developed ratios of geometric sequences which allowed him to successfully compute tables of numbers that could be added together in order to perform a multiplication or a division. In modern mathematical terms, he was describing the addition and subtraction of *exponents* of a particular *base* number, which in turn, is equivalent to multiplication and division. It is interesting to note that his development of logarithms came at a time before the development of the concept of exponents and exponential notation (b^x, say). Nonetheless, by creating accurate tables of logarithms and producing scales that were proportional to their values, the invention of the slide rule followed within a few short years. Its user was able to perform quick multiplication and division calculations with sufficient accuracy for a wide variety of computational applications.

Exponents and Powers

Our present introduction to *the logarithm* continues with a review of the operation of raising a number to a power. For example, let's continue with the simple operation of multiplying the number 2 with itself some p number of times. We call this "raising" 2 to the power of p, written "2^p." If p =2, then we say that the *square* of 2 , or 2 to the power 2, is $2^2 = 2 \times 2 = 4$. For $p = 3$, the *cube* of 2, or 2 to the power 3, is $2^3 = 2 \times 2 \times 2 = 8$. Clearly 2 to the first power ($p = 1$) is simply $2^1 = 2$. For completeness we must also define the operation of 2 to the power zero; if 2^2 is half of 2^3, and 2^1 is half of 2^2, then 2^0 is half of 2^1.; and half of two is just one. Thus, for $p = 0$, two to the power *zero* is $2^0 = 1$. (In fact, by the same argument, any real number to the power of zero is equal to one.)

Let's plot the results of taking 2^p for the values of $p = 0, 1, 2, 3, 4, 5$:



The next question is, what would be 2 raised to some power that is not an integer? That is, what about $2^{0.5}$? $2^{2.25}$? $2^{3.659}$? What is the meaning of such an operation? Might the results of such operations "line up" with the values plotted above?

To proceed, let's look at a couple of properties of multiplying numbers an integer number of times. For example, take two numbers which we write as powers of 2 – call them 2^a and 2^b. If we multiply these two numbers together we arrive at 2^{a+b}. For example,

$$2^2 \times 2^3 = (2 \times 2) \times (2 \times 2 \times 2) = 2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 2^{2+3}.$$

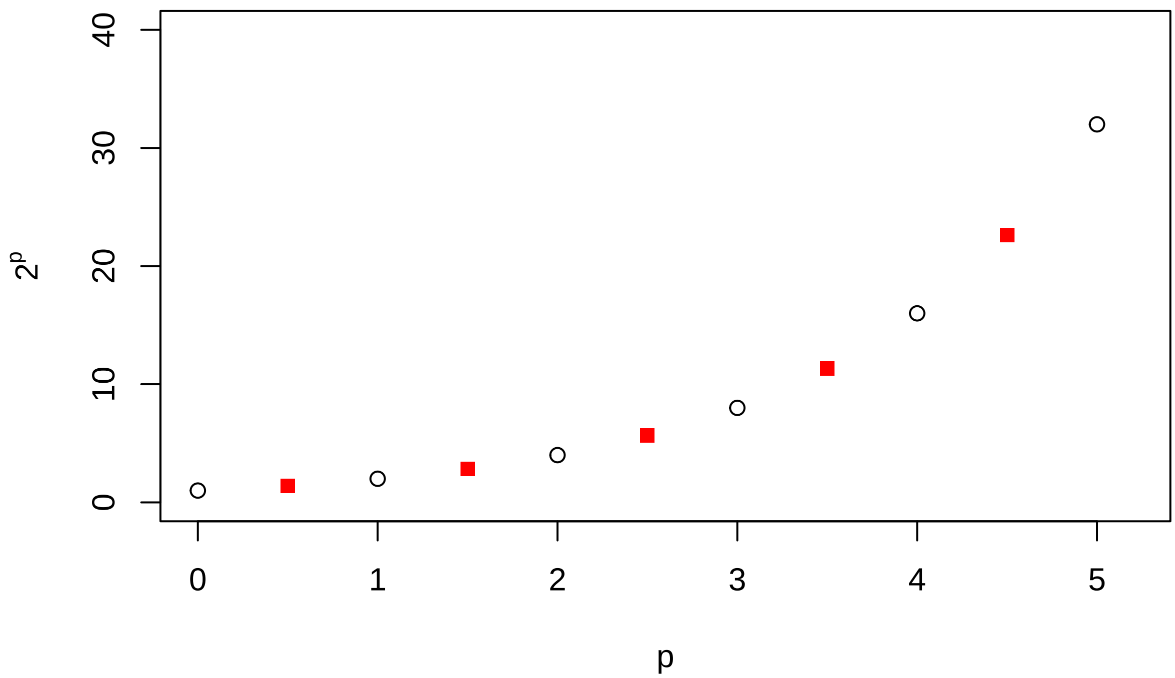
Additionally, if we take one of our numbers that is a power of 2 – say 2^a – and raise that to another integer power – say, a value of b – the result will be $(2^a)^b = 2^{a \times b}$; for example:

$$(2^2)^3 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) = (2 \times 2 \times 2 \times 2 \times 2 \times 2) = 2^6 = 2^{2\times3}.$$

In summary,

- $c^a \times c^b = c^{a+b}$, and
- $(c^a)^b = c^{a \times b}$.

If we accept these operations as general rules, then we can determine the values of our three operations that we have asked about. First, our rules tell us that $2^{0.5} \times 2^{0.5} = 2^1 = 2$. Hence, $2^{0.5}$ is the number that, when multiplied by itself, yields 2. We call this the *square root* of two, and we know its value: $\sqrt{2} \approx 1.414$. Thus, we can interpret $2^{0.5} = \sqrt{2} \approx 1.414$. Good approximations to the square roots of numbers can be found by various iterative techniques, which are often taught in Middle School mathematics classes. And now that we have a value for $2^{0.5}$, we can see from our basic rules above that $2^{1.5} = 2 \times 2^{0.5}$ and $2^{2.5} = 2^2 \times 2^{0.5}$ and so forth, and so it is easy to fill in more points on our graph (red squares):



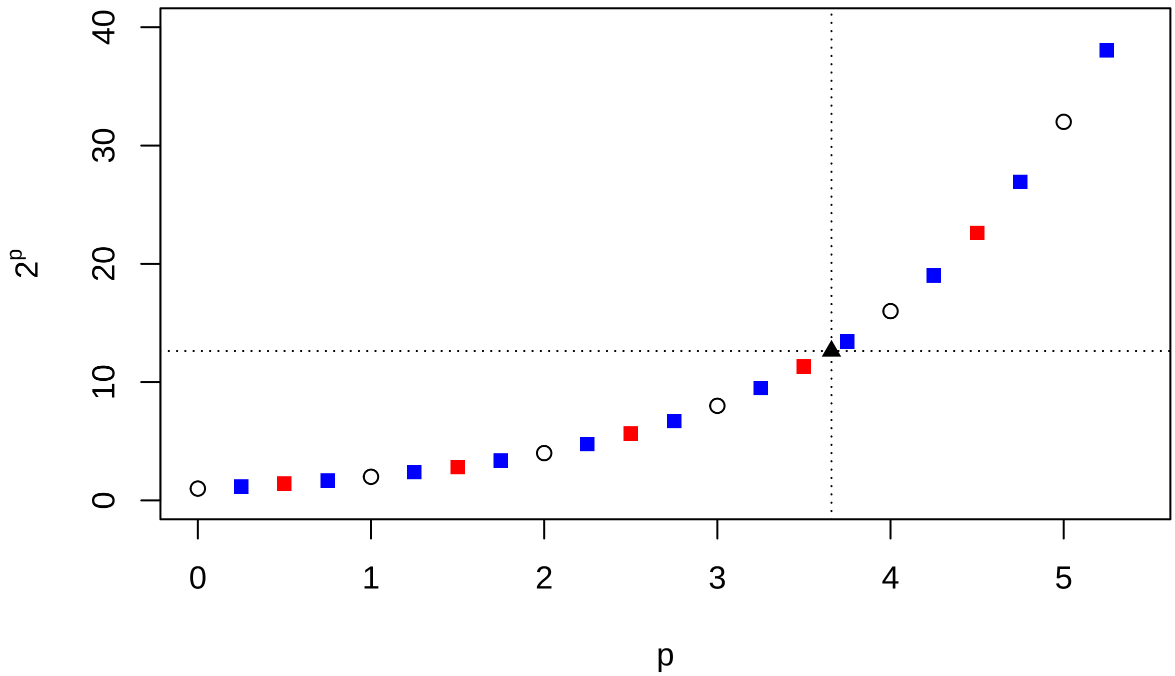
Next, what about $2^{2.25}$? Again using our new rules above, we see that

$$2^{2.25} = 2^{2+0.25} = 2^2 \times 2^{0.25} = 2^2 \times 2^{\frac{1}{4}}.$$

We call $2^{\frac{1}{4}} = \sqrt[4]{2}$ the "*fourth root* of 2" since, according to our rules,

$$2^{\frac{1}{4}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{4}} = 2.$$

But through our rules of exponents we also see that $2^{\frac{1}{4}} = (2^{\frac{1}{2}})^{\frac{1}{2}} = \sqrt{\sqrt{2}} \approx \sqrt{1.414} \approx 1.189$. Thus, finally, $2^{2.25}$ will be $2^2 = 4$ times this number, or 4.757. And, as before, knowing the value of $2^{0.25}$, we can also easily compute $2^{1.25} = 2 \times 2^{0.25}$, $2^{3.25} = 2^3 \times 2^{0.25}$ and so forth. In addition, $2^{0.75} = 2^{0.5} \times 2^{0.25} = 1.6818$, and thus $2^{1.75} = 2 \times 2^{0.75}$, $2^{2.75} = 2^2 \times 2^{0.75}$, and so forth. Now, we can fill in our plot even further (blue squares):



We see that the circles and the red and blue squares follow a general trend. So far we've been lucky to use numbers and operations (square roots) that we know well and understand. Continuing in this way, we can take further roots and powers and fill in the plot to whatever level necessary. So what about our third example, $2^{3.659}$? Looking at our graph above, if we follow the vertical line at $p = 3.659$ up to our group of points, we might expect $2^{3.659}$ to have a value of about 13 or so, as read off the left-hand scale.

Following our rules above, we can write $2^{3.659}$ as $2^3 \times 2^{0.659} = 8 \times 2^{0.659}$. So one way (there are many) to arrive at the answer is to re-write $2^{3.659}$ as $2^3 \times (2^{1/1000})^{659}$. But $2^{1/1000} = ((2^{1/10})^{1/10})^{1/10}$, and $2^{1/10} = (2^{1/5})^{1/2}$. Hence,

$$2^{3.659} = 2^3 \times \left[(((((2^{1/5})^{1/2})^{1/5})^{1/2})^{1/5})^{1/2}\right]^{659} = 2^3 \times \left[\sqrt{(\sqrt{(\sqrt{2^{1/5}})^{1/5}})^{1/5}}\right]^{659}.$$

If we are able to find the fifth root of a number, as well as the square root of a number, then we can compute $2^{3.659}$. That is, we can find the fifth root of 2 and take the square root of this result. Then take this answer and repeat the process two more times. Finally, we take this result, multiply it with itself 658 times, and end by multiplying by 8. This might take quite awhile by hand, but can be done easily on a modern personal computer:

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## [1] (2^3)*(sqrt(sqrt(sqrt(2^(1/5) )^(1/5) )^(1/5) )^659) = 12.6319021872098
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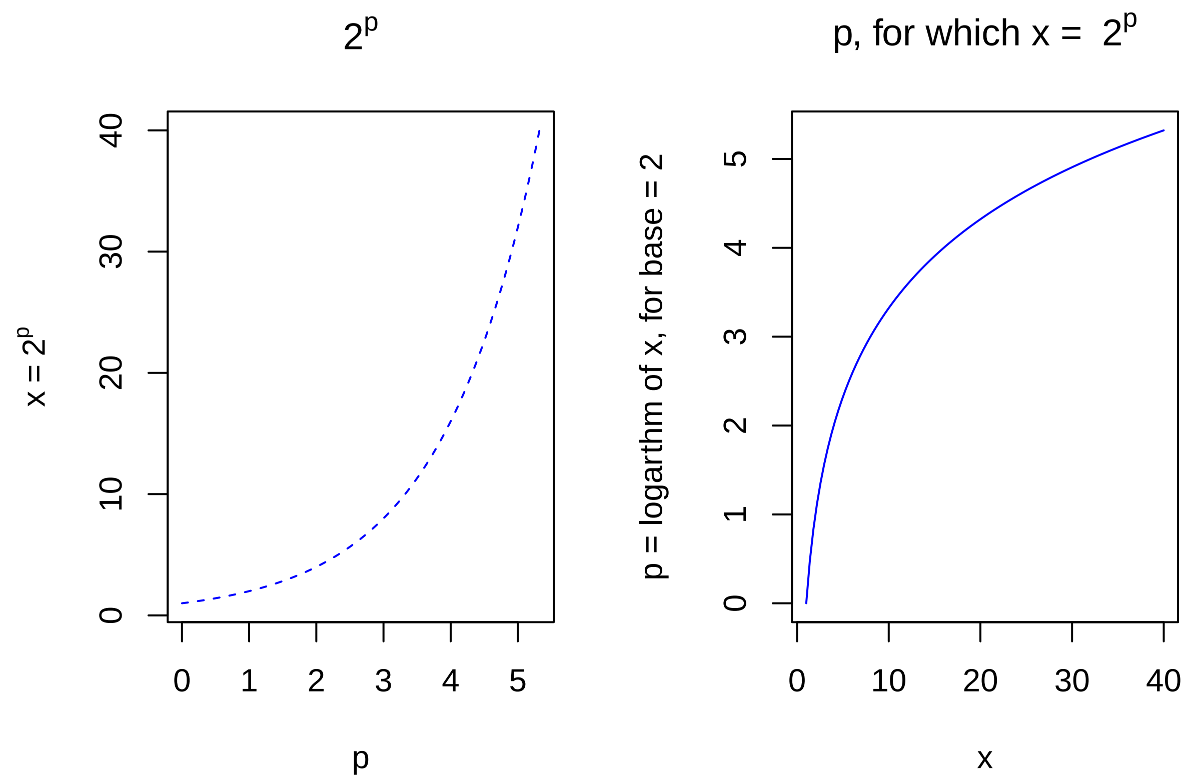
The value of $2^{3.659}$ =12.6319 is included on our plot above as a black diamond and, again, falls into the pattern of our other computations. If we continue in this vein, in principle we could fill in an entire curve and hence for any chosen value of p, integer or non-integer, we can compute a value for the operation 2^p to some desired accuracy.

Empowered with this new information, problems now can be posed in reverse. We can imagine filling in our graph until a continuous curve appears, and then take this curve and switch the axes as shown below. The point of this switching of the axes is to emphasize that any number x can be represented by the number 2 raised to a particular power p:



The concept of a *logarithm* is to chose a base (2, in our example here) and ask the question, "Given a general number x, to what power must we raise the base in order to obtain this number?" The answer – the exponent p – is called the *logarithm* of x for that particular base. Although we used 2 as our base number above, the choice of "2" is arbitrary. The same procedures could be used for any base we choose such as 3 or 10, and includes the use of non-integers or even irrational numbers such as $\pi = 3.141592\ldots$ The only caveats at the moment are that the base number be greater than zero (zero raised to any power is zero; and negative numbers complicate things) and it cannot be one. (One raised to any power is just one.)

So why go through all of the trouble of finding logarithms? Suppose one wanted to multiply two multi-digit numbers, x and y. Multiplying two numbers such as $x = 28718$ and $y = 81793$ could take quite a while and be prone to errors if not performed very carefully. With a table of values of logarithms for our chosen base (here, base = 2), we can look up the values of the *logarithms* p and q for the numbers x and y that satisfy $x = 2^p$ and $y = 2^q$. Next, we reason that $x \times y = 2^{p+q}$. While multiplying the numbers x and y may take take awhile by hand, the summation of p and q can be performed much more quickly and easily (and, typically, with much less chance for error). We then look in our table for the number that has the total of $p + q$ as its logarithm, and this would be our final answer for $x \times y$.

The realization that the multiplication of numbers can be equated to the addition of so-called *logarithms* was Napier's major achievement. If logarithms of real numbers could be tabulated to sufficient accuracy, then complicated multiplication and division problems could be reduced to much simpler addition and subtraction calculations. In Napier's day, this was particularly important in navigation calculations performed at sea, for instance. Now, such calculations could be performed much more quickly and accurately. One didn't need to spend large amounts of time multiplying large numbers long-hand using pen and paper. Rather, logarithms could be looked up in a book of numerical tables and added together to get a result. Napier and his English colleague, Henry Briggs, produced detailed tables of common logarithms within just a few years of their mathematical development. And the ability – even in the 1600's – to compute the values of logarithms of numbers to sufficient accuracy led almost immediately to the invention of the slide rule. A systematic procedure for calculating the logarithm of an arbitrary number is left to the [next chapter][Computing Logarithms]. Meanwhile, in the remaining sections of the present chapter, we will assume that values of logarithms are available to us as we investigate their various properties and explore their use in performing basic calculations.

Logarithms and Log Scales

Consider an arbitrary positive real number x. As we just discussed, we can express x in terms of a *base* number – say, b – raised to a power p:

$$x = b^p.$$

In standard mathematical language we say that the *logarithm* of the number x is the exponent p:

$$\log_b x = p.$$

The above expression is read, "The log, base b, of x is equal to p." This definition leads to the following identity:

$$x = b^{\log_b x}.$$

In mathematical terms we say that the function $\log_b x$ is the *inverse* of the function b^x – if we start with x and find b^x, and then take the logarithm using base b of the resulting number, we get x back again:

$$\log_b b^x = x.$$

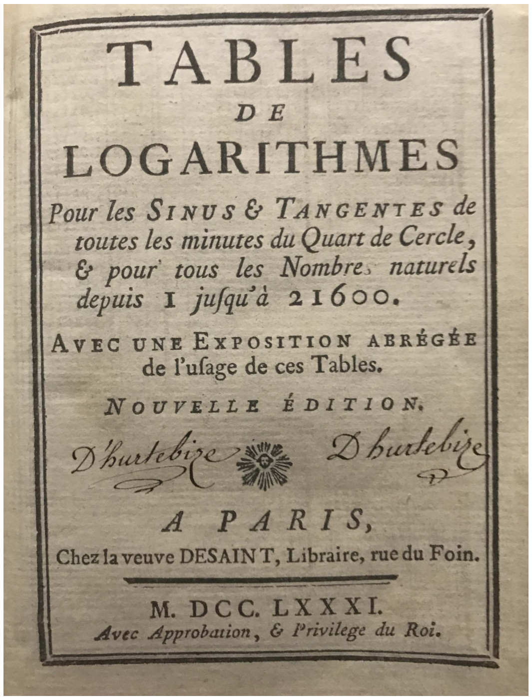
In our earlier example we used 2 as a base, but the most common base to use, which is what typical slide rules have been based upon (no pun intended), is $b = 10$, or "Base 10": $x = 10^p$. The standard is simply to define *the* logarithm – or, the ***common logarithm*** – as that using Base 10:

$$\text{if} \quad x = 10^p \quad \longrightarrow \quad \log x \equiv \log_{10} x = p.$$

In other words, ask, "To what power must you raise the number 10 in order to obtain the number x?" The answer, p, is the *logarithm* of x (simply called "log of x"). The convention is to write "$\log x$" when the *base* being used is 10. If it were another base, say 2, then the expression would be written $\log_2 x$.

We saw earlier that any number raised to the power of "zero" is by definition "one". Hence, log $1 = 0$. That is, $1 = 10^0$. The logarithm of a number greater than 1 will be a positive number. For example, $10 = 10^1$ and $100 = 10^2$, so $\log 10 = 1$ and $\log 100 = 2$. The logarithm of a number less than 1 will be negative: $0.1 = 10^{-1}$ and $0.01 = 10^{-2}$, hence $\log 0.1 = -1$ and $\log 0.01 = -2$. What about other, more general numbers? As we did before for our "Base 2" example, we can take the square root of 10, which is $10^{1/2} = 10^{0.5} = \sqrt{10}$ $= 3.1623$, and hence we can infer that $\log 3.1623 = 0.5$. It can also be verified that $10^{0.30103}$ yields 2.0000. Thus, $\log 2$ is approximately equal to 0.30103. Precise values of the common logarithms of numbers can be calculated by using calculus, finite difference techniques, or other methods, and their values have been computed and tabulated with various degrees of accuracy over the past centuries. One such method for computing logarithms is presented in the [next chapter][Computing Logarithms]. Books containing tables of values of logarithms were painstakingly produced and published for use in the types of calculations that will be described below; these books are also collectible and can be found in many used book stores.

A table of logarithms from the year 1781.[2]



TABLES
DE
LOGARITHMES

Pour les SINUS & TANGENTES *de toutes les minutes du Quart de Cercle, & pour tous les Nombres naturels depuis* 1 *jusqu'à* 21600.

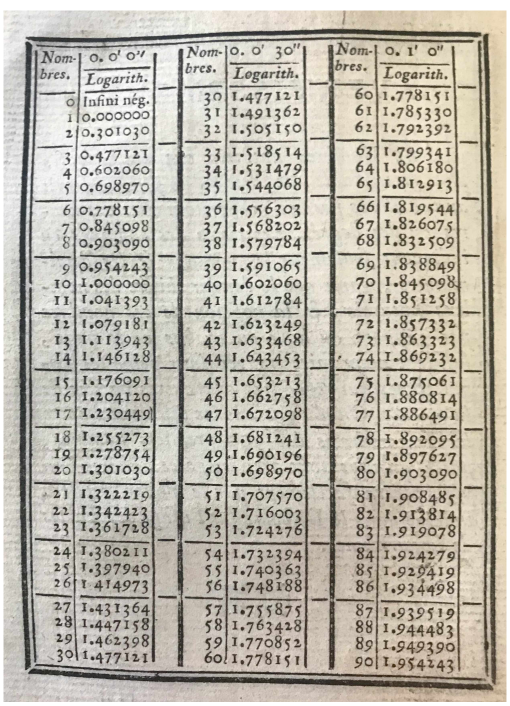
AVEC UNE EXPOSITION ABRÉGÉE de l'usage de ces Tables.

NOUVELLE ÉDITION.

D'hurtebize *D'hurtebize*

A PARIS,
Chez la veuve DESAINT, Libraire, rue du Foin.

M. DCC. LXXXI.
Avec Approbation, & Privilege du Roi.



Nombres.	0. 0' 0" Logarith.	Nombres.	0. 0' 30" Logarith.	Nombres.	0. 1' 0" Logarith.
0	Infini nég.	30	1.477121	60	1.778151
1	0.000000	31	1.491362	61	1.785330
2	0.301030	32	1.505150	62	1.792392
3	0.477121	33	1.518514	63	1.799341
4	0.602060	34	1.531479	64	1.806180
5	0.698970	35	1.544068	65	1.812913
6	0.778151	36	1.556303	66	1.819544
7	0.845098	37	1.568202	67	1.826075
8	0.903090	38	1.579784	68	1.832509
9	0.954243	39	1.591065	69	1.838849
10	1.000000	40	1.602060	70	1.845098
11	1.041393	41	1.612784	71	1.851258
12	1.079181	42	1.623249	72	1.857332
13	1.113943	43	1.633468	73	1.863323
14	1.146128	44	1.643453	74	1.869232
15	1.176091	45	1.653213	75	1.875061
16	1.204120	46	1.662758	76	1.880814
17	1.230449	47	1.672098	77	1.886491
18	1.255273	48	1.681241	78	1.892095
19	1.278754	49	1.690196	79	1.897627
20	1.301030	50	1.698970	80	1.903090
21	1.322219	51	1.707570	81	1.908485
22	1.342423	52	1.716003	82	1.913814
23	1.361728	53	1.724276	83	1.919078
24	1.380211	54	1.732394	84	1.924279
25	1.397940	55	1.740363	85	1.929419
26	1 414973	56	1.748188	86	1.934498
27	1.431364	57	1.755875	87	1.939519
28	1.447158	58	1.763428	88	1.944483
29	1.462398	59	1.770852	89	1.949390
30	1.477121	60	1.778151	90	1.954243

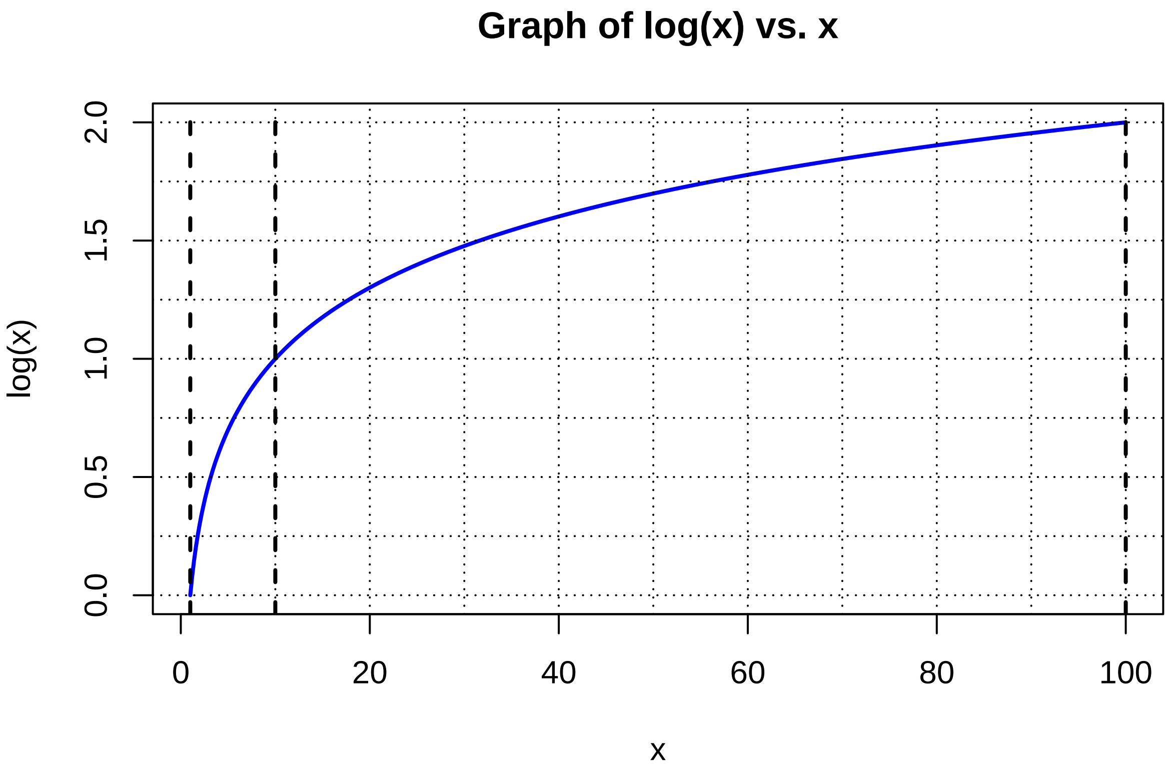
[2] *Tables de Logarithmes*, Chez veuve DESAINTE, Libraire, rue du Foin, Paris, 1781.

Modern electronic computer languages have built-in algorithms that can produce logarithms of numbers to high accuracy. The following is a table of the Base 10 common logarithms of integer numbers between 1 and 100, similar to what can be found in a book of logarithms, but here generated by a modern personal computer:

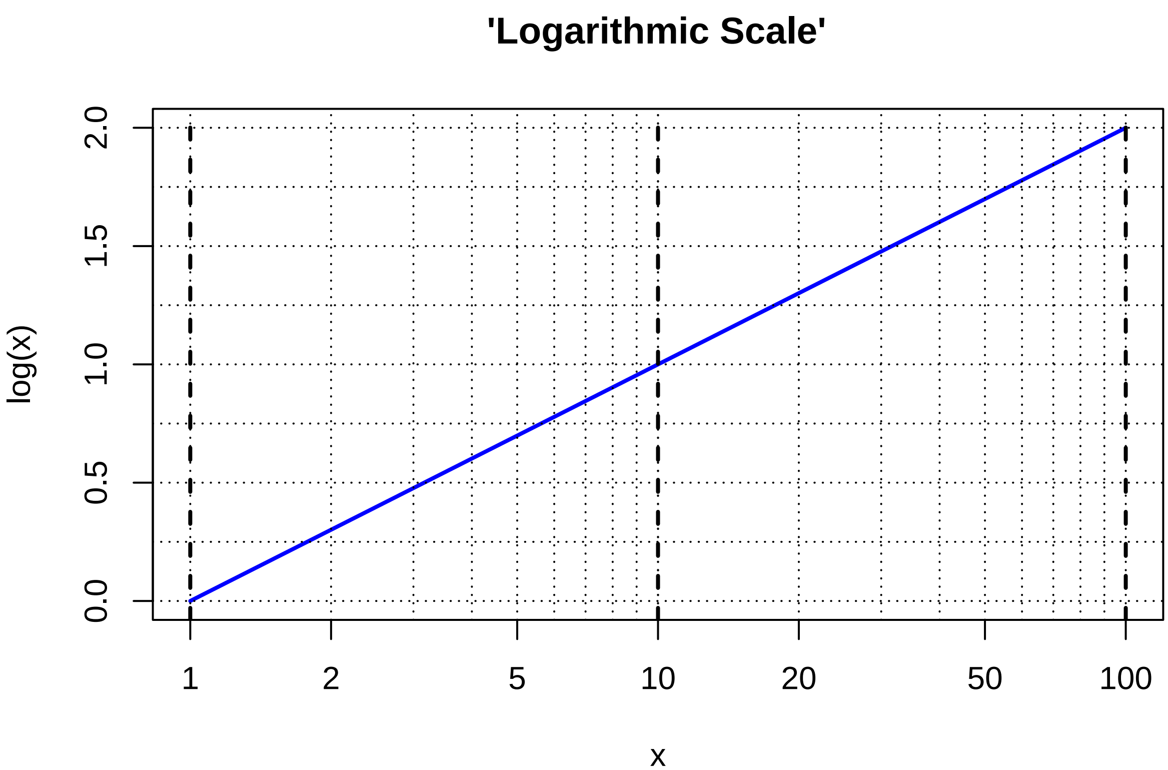
Table of Common Logarithms

x	log x	x	log x	x	log x	x	log x
1	0.00000	26	1.41497	51	1.70757	76	1.88081
2	0.30103	27	1.43136	52	1.71600	77	1.88649
3	0.47712	28	1.44716	53	1.72428	78	1.89209
4	0.60206	29	1.46240	54	1.73239	79	1.89763
5	0.69897	30	1.47712	55	1.74036	80	1.90309
6	0.77815	31	1.49136	56	1.74819	81	1.90849
7	0.84510	32	1.50515	57	1.75587	82	1.91381
8	0.90309	33	1.51851	58	1.76343	83	1.91908
9	0.95424	34	1.53148	59	1.77085	84	1.92428
10	1.00000	35	1.54407	60	1.77815	85	1.92942
11	1.04139	36	1.55630	61	1.78533	86	1.93450
12	1.07918	37	1.56820	62	1.79239	87	1.93952
13	1.11394	38	1.57978	63	1.79934	88	1.94448
14	1.14613	39	1.59106	64	1.80618	89	1.94939
15	1.17609	40	1.60206	65	1.81291	90	1.95424
16	1.20412	41	1.61278	66	1.81954	91	1.95904
17	1.23045	42	1.62325	67	1.82607	92	1.96379
18	1.25527	43	1.63347	68	1.83251	93	1.96848
19	1.27875	44	1.64345	69	1.83885	94	1.97313
20	1.30103	45	1.65321	70	1.84510	95	1.97772
21	1.32222	46	1.66276	71	1.85126	96	1.98227
22	1.34242	47	1.67210	72	1.85733	97	1.98677
23	1.36173	48	1.68124	73	1.86332	98	1.99123
24	1.38021	49	1.69020	74	1.86923	99	1.99564
25	1.39794	50	1.69897	75	1.87506	100	2.00000

Taking a few examples, the values in the table tell us that the number 7 is equivalent to $10^{0.8451}$, $12 = 10^{1.07918}$, $63 = 10^{1.79934}$, and $100 = 10^2$. In the following curve the value x is read on the horizontal axis and its logarithm is read on the vertical axis:



The heavy vertical dashed lines in the plot above are at values of $x = 10^0$, 10^1, and 10^2, and we see that they cross the blue curve at values of 0, 1, and 2 on the left-hand scale. Now suppose that we *stretch* the numerical scaling of the horizontal axis to create a linear relationship to the values of logarithms along the vertical axis.



The special scaling along the x axis shown in the final figure above is often referred to as a “log scale” and this is what is found on the standard scales of a slide rule. The figure shows that with this particular scaling on the slide rule, the distance from the number 1 to the number x is directly proportional to the logarithm of the number x. Notice that the *distance* from 1 to 10 along the horizontal axis is the same as the *distance* from 10 to 100 (as would be the *distance* between 100 and 1000 or between 0.1 and 1, and so forth). Every factor of 10 changes the logarithm by one unit. One can see also that the logarithm of 20 (distance from 1 to 20) is the same as the logarithm of 10 (distance from 1 to 10) plus the logarithm of 2 (distance from 1 to 2).

Multiplication and Division

Suppose we have two numbers, x and y and we wish to multiply them together. If

$$x = 10^p \quad \text{and} \quad y = 10^q, \qquad \text{then} \qquad x \times y = 10^p \times 10^q = 10^{p+q}.$$

Thus we see that the logarithm of $x \times y$ will be the sum of the exponents p and q, which are themselves the logarithms of the original numbers. This gives the very important result:

$$\log(x \times y) = \log x + \log y.$$

Adding the logarithms of two numbers gives the logarithm of the *product* of the two numbers!

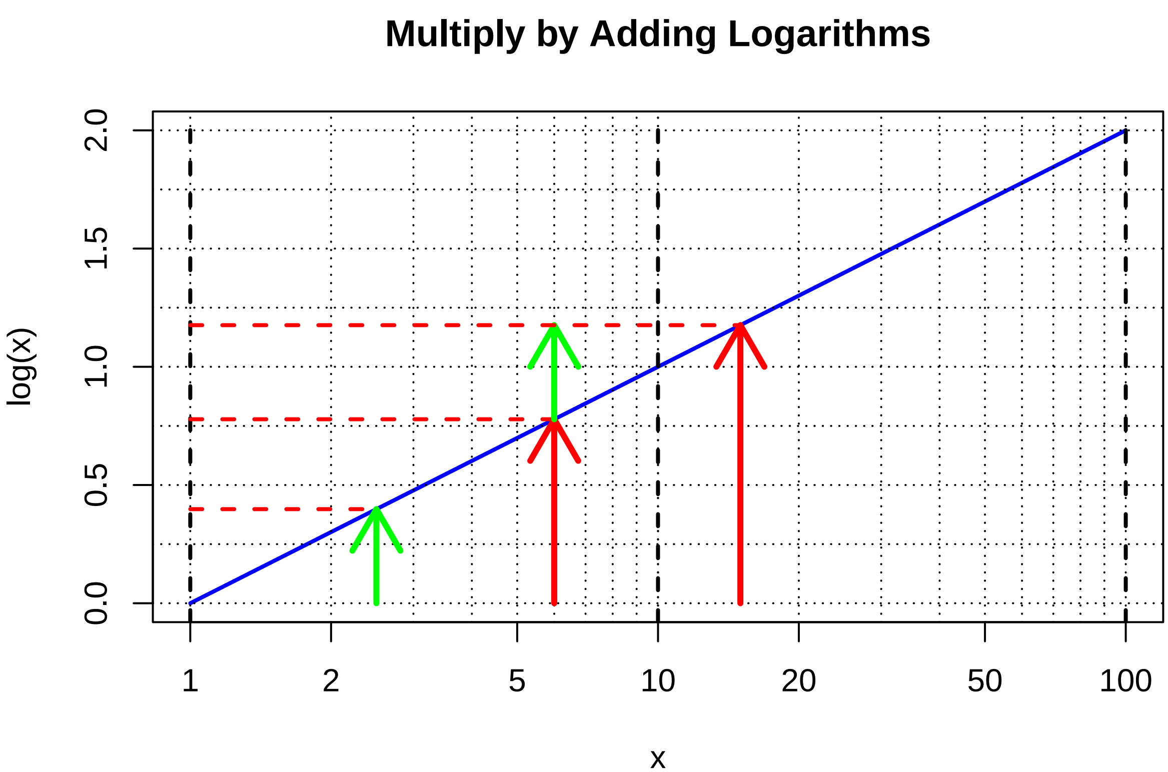
As for division, just remember that dividing by a number is equal to multiplying by that number's reciprocal:

$$x \div y = \frac{x}{y} = x \times \frac{1}{y} = 10^p \times \frac{1}{10^q} = 10^p \times 10^{-q} = 10^{p-q}.$$

So, subtracting the logarithms of two numbers gives the logarithm of their ratio:

$$\log\left(\frac{x}{y}\right) = \log x - \log y.$$

The basic operations of the slide rule involve adding and subtracting distances along the rule which are proportional to the logarithms of the numbers on the scales. Looking at our previous plot, we can see that *adding* the logarithms of two numbers, let's say 2.5 and 6, gives us the logarithm of their product – in this case, 15:



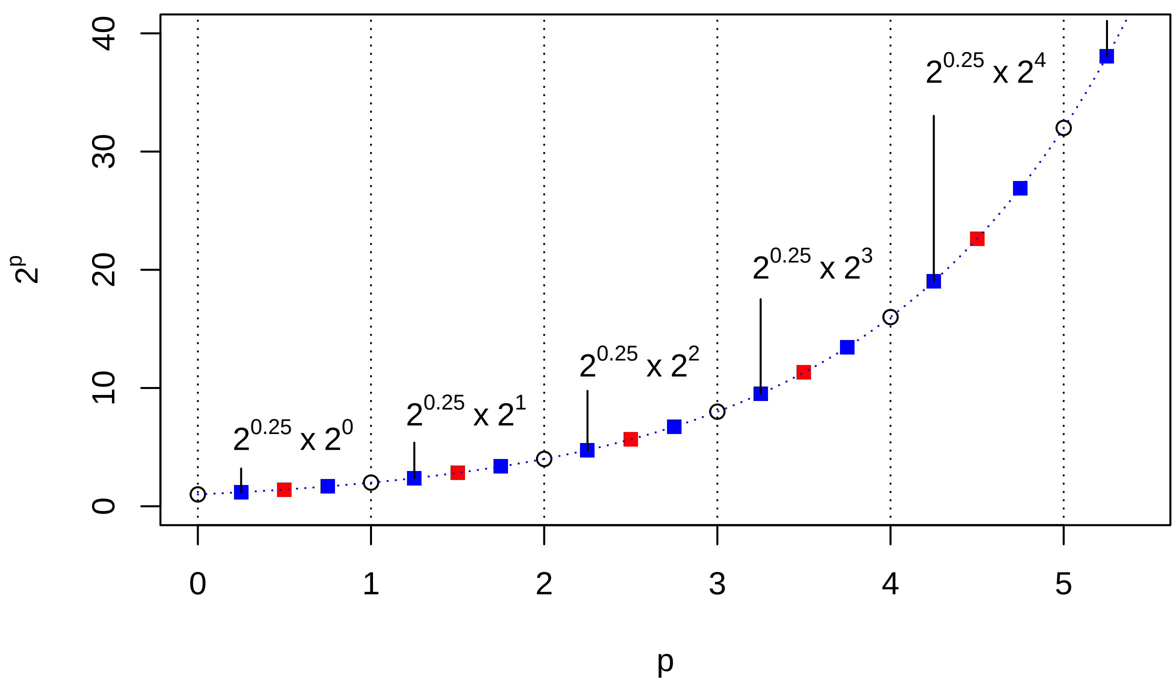
Keeping Track of Decimal Places

Consider the numbers 3.7, 37, and 370. Using our rules for logarithms, we see that each power of ten adds "one" to the value of the logarithm:

$$\begin{array}{ccccccccc} \log 3.7 & = & \log(3.7 \times 1) & = & \log 3.7 & + & \log 1 & = & \log 3.7 + 0 \\ \log 37 & = & \log(3.7 \times 10) & = & \log 3.7 & + & \log 10 & = & \log 3.7 + 1 \\ \log 370 & = & \log(3.7 \times 100) & = & \log 3.7 & + & \log 100 & = & \log 3.7 + 2 \end{array}$$

We can see a similar pattern when looking back at our table of logarithms above. For example, notice that $\log 6 = 0.7782$, while $\log 60 = 1.7782$. By keeping track of powers of 10, one only needs the values of the logarithms for numbers between 1 and 10 to be able to determine the logarithm of any other number outside that range.

As an illustration, recall the plot we made earlier using Base 2. The pattern between each integer value of p repeats; the corresponding point in each repeat section is just "2" times larger than the point in the previous section of the plot:



Hence, if we want to know the value of $2^{7.25}$, we only need to know the value of $2^{0.25}$ and then multiply this answer by 2^7 (= 128).

We can use any number as our base, and the same general rules will apply; for our Base 10 system of common logarithms, multiplying by factors of 10 simply means adding zeroes or moving decimal points. Hence, by keeping track of powers of ten, detailed tables of common logarithms of numbers between 1 and 10, which themselves will have values between 0 and 1, are sufficient to perform a fairly accurate general multiplication or division calculation, as we shall soon see.

For example, the table below gives the 3-place logarithms for numbers between 1 and 10 in increments of 0.1. For finer increments and for further accuracy of the logarithms, tables can take many pages of text. This led to the publication of books of significant length that contained detailed tables of logarithms.

x	log x	x	log x	x	log x	x	log x
1.0	0.000	3.3	0.519	5.6	0.748	7.9	0.898
1.1	0.041	3.4	0.531	5.7	0.756	8.0	0.903
1.2	0.079	3.5	0.544	5.8	0.763	8.1	0.908
1.3	0.114	3.6	0.556	5.9	0.771	8.2	0.914
1.4	0.146	3.7	0.568	6.0	0.778	8.3	0.919
1.5	0.176	3.8	0.580	6.1	0.785	8.4	0.924
1.6	0.204	3.9	0.591	6.2	0.792	8.5	0.929
1.7	0.230	4.0	0.602	6.3	0.799	8.6	0.934
1.8	0.255	4.1	0.613	6.4	0.806	8.7	0.940
1.9	0.279	4.2	0.623	6.5	0.813	8.8	0.944
2.0	0.301	4.3	0.633	6.6	0.820	8.9	0.949
2.1	0.322	4.4	0.643	6.7	0.826	9.0	0.954
2.2	0.342	4.5	0.653	6.8	0.833	9.1	0.959
2.3	0.362	4.6	0.663	6.9	0.839	9.2	0.964
2.4	0.380	4.7	0.672	7.0	0.845	9.3	0.968
2.5	0.398	4.8	0.681	7.1	0.851	9.4	0.973
2.6	0.415	4.9	0.690	7.2	0.857	9.5	0.978
2.7	0.431	5.0	0.699	7.3	0.863	9.6	0.982
2.8	0.447	5.1	0.708	7.4	0.869	9.7	0.987
2.9	0.462	5.2	0.716	7.5	0.875	9.8	0.991
3.0	0.477	5.3	0.724	7.6	0.881	9.9	0.996
3.1	0.491	5.4	0.732	7.7	0.886	10.0	1.000
3.2	0.505	5.5	0.740	7.8	0.892	NA	NA

One can immediately see how logarithms can be used to multiply numbers. For example, notice from our table that the logarithm of 1.5 is 0.176 and the logarithm of 3.6 is 0.556. The sum of the logarithms is 0.732, which we can see from the table is the logarithm of $5.4 = 1.5\times\ 3.6$. Clearly many other examples can be found just from this simple table. Now suppose we wanted to multiply $150\times\ 3.6$. We don't need to have the logarithm of 150 in our table – the logarithm of 150 will just be 2 + the logarithm of 1.5, or 2.176. The result will just be 10^2 times the result for $1.5\times\ 3.6$, or 540. By keeping track of powers of ten, calculations can be performed by just using logarithms of numbers between 1 and 10.

Scientific Notation

Numbers encountered in many science and engineering problems can become quite large or quite small, depending upon the units of measure being used. Writing numbers in what is called *scientific notation* makes keeping track of decimal points and orders of magnitude (factors of ten) simpler, as well as making computations using logarithms more straightforward. With a choice of 10 as our base, we can write any positive number as a number between 1 and 10, times 10 raised to an integer power. A couple of examples help:

$$5827 = 5.827 \times 10^3; \qquad 0.0365 = 3.65 \times 10^{-2}$$

If we take the logarithm of our first number, as an example, and use our rules of multiplication, we see that

$$\log 5827 = \log(5.827 \times 10^3) = \log 5.827 + \log 10^3 = \log 5.827 + 3.$$

Likewise,

$$\log 0.0365 = \log 3.65 + \log 10^{-2} = \log 3.65 - 2.$$

Remember that 10^0 is just 1, and that the log of zero doesn't exist – there is no finite power to which one can raise the number 10 in order to produce zero exactly. Zero is just zero, and anything times zero is also zero.

As an easy example for illustration, consider the following multiplication : 1500×0.08. Using our prior table of 3-place logarithms, we see that

$$\begin{aligned}
r &= 1500 \times 0.08 \\
&= 1.5 \times 10^3 \quad \times \quad 8 \times 10^{-2} \\
\\
\longrightarrow \quad \log r &= \log 1.5 + 3 \quad + \quad \log 8.0 - 2 \\
&= \log 1.5 + \log 8.0 \quad + \quad (3-2) \\
&= \ \ 0.176 + 0.903 \quad + \quad (3-2) \\
&= 1.079 + 1 \\
&= 0.079 + 2 \\
&= \log 1.2 + 2 \qquad \text{(from the table)} \\
\\
\therefore \quad r &= 120
\end{aligned}$$

Scientific notation became a popular way of expressing numbers being used in calculations, due to its emphasis of the logarithmic approach. When it was necessary to multiply and divide a long string of numbers, it was easier to write each number in scientific notation, add or subtract the relevant logarithms of each, and keep track of the "factors of ten" to determine the final placement of the decimal point.

Short Examples

To get a feel for using logarithms in calculations, let's perform a few examples. We will suppose that we have a table of logarithms already prepared, where the number x is tabulated for the range $1 \le x \le 10$, sub-divided into intervals such as 10-ths, or 100-ths, say, and the logarithms $\log x$ are entered into the table with a certain accuracy, such as 5 decimal places, and are in the range $\log 1 \le \log x \le \log 10$, or equivalently, $0 \le \log x \le 1$.

For example, suppose we want to compute 38900×0.792. We start by re-writing the problem as follows:

$$38900 \times 0.792 = 3.89 \times 10^4 \times 7.92 \times 10^{-1} = 3.89 \times 7.92 \times 10^3.$$

Then, taking the logarithm of this result,

$$\log(3.89 \times 7.92 \times 10^3) = \log 3.89 + \log 7.92 + 3$$

Then, from a table of 5-place logarithms we find that

$$\log(38900 \times 0.792) = 0.58995 + 0.89873 + 3 = 1.48868 + 3 = 0.48868 + 4.$$

Next, we find the number between 1 and 10 that has the logarithm 0.48868. From our table of logarithms we find that the number with logarithm 0.48868 is, to five significant figures: 3.0809. The FINAL result of our multiplication is hence $3.0809 \times 10^4 = 30809$. Check (via computer):

```
## [1] 38900 * 0.792  =  30808.8
```

As we can see, the answer is correct, though its accuracy from the logarithmic approach will depend upon the accuracy of the logarithm values used. Here, our answer with 5-place logarithms is good to 6 parts per million. Such accuracy might be important in some applications, but might be overkill in others – do you need to know the height of that tree in your backyard to within 50 microns?

As another example, let's divide 245 by 5960. To start, write

$$\frac{245}{5960} = \frac{2.45 \times 10^2}{5.96 \times 10^3} = \frac{2.45}{5.96} \times 10^{-1}.$$

Then, proceeding as before, we get

$$\log(245/5960) = \log 2.45 - \log 5.96 - 1 = 0.38917 - 0.77525 - 1 = -0.38608 - 1$$

But our logarithm tables are for numbers that are greater than one. So, we add 1 and subtract 1 to the above to get

$$\log(245/5960) = -0.38608 - 1 = 1 - 0.38608 - 2 = 0.61392 - 2$$

which gives us a positive "remainder" (often called the *mantissa*) between 0 and 1, plus an integer (often called the *characteristic*) which in this case happens to be negative.

Then, from our logarithm table, we find that 0.61392 is the logarithm of 4.111. But the "-2" tells us to move the decimal point two places to the left So, finally, we see that 245/5960 must be equal to 0.04111. By computer we find that 245/5960 = 0.041107.

Taking two numbers that we used earlier, let $x = 5827$, and $y = 0.0365$. The product is $x \times y = 212.6855$, which can be verified by straightforward multiplication using pencil and paper, if you want to take the time. But now let's compute this product using logarithms. Here, we'll use a computer to do the work,[3] where the function `log10(a)` below produces the Base 10 logarithm of the number `a` to a high degree of precision:

```
x = 5827
y = 0.0365
logx  = log10(x)
logy  = log10(y)
```

The output:

```
## [1] logx = 3.76544501809015 ; logy = -1.43770713554353 .
```

```
## [1] logx + logy = 2.32773788254662 .
```

```
## [1] 10^( 2.32773788254662 ) = 212.6855 .
```

The number with logarithm 2.32774 is 212.6855.

The computer example above illustrates well the usefulness of having values of logarithms accurate to many decimal places. Of course, with a computer one doesn't need to perform multiplications using logarithms. But prior to computers, using tables of logarithms was a standard approach to perform multiplications and divisions with high accuracy, and was taught in schools until the late 1970s.

Before electronic computing devices, one could have used a book of tables to look up the appropriate logarithms and then add them together to perform a "multiplication". Then, one would take that result and find in the tables the number that had that value as its logarithm; this would be the final answer, with *the computer* (the one doing the computing) having to keep track of the powers of ten. This would be repeated as needed to multiply and/or divide a series of numbers.

The slide rule provides access to the values of logarithms (to a certain accuracy) and enables quick manipulations of them for mathematical calculations without the need to write down each of the steps and without looking up numbers in tables. The slide rule greatly sped up the process of performing long calculations, when the final results were only required to be accurate to a few digits. We will come back to this procedure when we get to [Slide Rule ABC's and D's].

[3] Throughout this text we are using the **R** programming language. See [@R-base].

Raising Numbers to Powers

Starting with our first equation in this chapter, we had $x = b^p$ and we said that the logarithm of x is $\log_b x = p$. Now suppose we take x and raise it to the power of r,

$$x^r = (b^p)^r = b^{p \times r},$$

and then take the logarithm:

$$\log_b x^r = p \times r.$$

Since $p = \log_b x$, we have the general result:

$$\log_b x^r = r \times \log_b x.$$

To raise a number x to some arbitrary power r, one would take the logarithm of x, find the product of that number times r, and then find the number whose logarithm is equal to that product.

Suppose we wish to compute $y = 7^{2/5}$. Finding the logarithm of each side, we get

$$\log y = \log 7^{2/5} = 2/5 \times \log 7 = 0.4 \times 0.84510 = 0.33804$$

The number whose log is 0.33804 is then looked up in a table, say, and found to be $y = 2.1779$.

As a check, we can compute $y^{5/2} = (\sqrt{y})^5$ on a computer, which should give us "7" to good accuracy:

```
## [1] sqrt(2.1779)*sqrt(2.1779)*sqrt(2.1779)*sqrt(2.1779)*sqrt(2.1779)  =  7.0000
```

For the case where the exponent is a negative number, the above relationship shows that division is implied, as the resulting logarithm will be subtracted in any ongoing calculation:

$$\log \frac{1}{x^r} = \log x^{-r} = -r \times \log x.$$

This also emphasizes that a number less than one will have a negative logarithm. For instance, suppose the number $u = x^r$, where x and r are each greater than one. Then $u > 1$. Now the number $v = 1/u < 1$ and its logarithm will be $\log v = \log x^{-r} = -r \cdot \log x < 0$. Consider, for example, $\log \frac{1}{100} = \log 0.01 = \log 10^{-2} = -2 \log 10 = -2$.

Another Example

Let's compute the volume of a sphere of radius 4.58 inches. This time, we will use 3-place logarithms (i.e., logarithms to three decimal places).

The formula for the volume of a sphere is

$$V = \frac{4}{3}\pi r^3.$$

So, using the rules we have just learned,

$$\begin{aligned}
\log V &= \log 4 - \log 3 + \log \pi + \log 4.58^3 \\
&= \log 4 - \log 3 + \log \pi + 3 \times \log 4.58 \\
&= 0.602 - 0.477 + 0.497 + 3 \times (0.661) \\
&= 2.605 = 0.605 + 2
\end{aligned}$$

The number with logarithm 0.605 is 4.027. The "2" tells us to multiply by 100. So, this gives the final value of the volume, $V = 402.7$ cubic inches. (Again, check with computer: 402.4250839 cubic inches. We have about a 0.07% error in our answer, using 3-place logarithms.) Of course, if our knowledge of the radius is only to three digits as given above, then no more than the first three digits in the answer should be of any significance. In real life, of course, one must always perform an appropriate *error analysis* of a result, but that is beyond the scope of our present tutorial.

The above example is a very common operation in science, engineering, and many industrial settings – the sequential multiplication and division of a series of numbers. Having a device, like the slide rule, that has logarithmic scales built in can allow the user to very quickly perform such operations without having to resort to books of tables of logarithms and writing down intermediate results.

Change of Base

If one has knowledge of the logarithm of a number for a particular base, say b, then the logarithm of the same number relative to a different base, say a, can be found *via* the following argument. If

$$x = a^p$$

then taking the logarithm using base b on both sides,

$$\log_b x = \log_b a^p = p \times \log_b a$$

using the rule we discussed earlier. But note that p is the logarithm of x using base a, and so

$$\log_b x = \log_a x \times \log_b a.$$

Thus, a switch from base b to base a is performed by

$$\log_a x = \log_b x / \log_b a.$$

More than just an interesting tidbit, this is actually an important result. It tells us that we do not need hundreds of tables of logarithm values for the vast number of different bases that might be of interest in different situations. We actually only need one table of values. If we have the logarithms tabulated for one base, we can easily compute values for any other base, as needed. From our historical path, and through a bit of evolution, we have become a "Base 10" civilization, and so logarithms using that base are chosen as the *common* logartihms. The development of the *calculus* and of functional analysis in mathematics shows that using the constant $e = 2.718\ldots$ is a natural choice for describing exponential growth and decay; logarithms with e as their base are called *natural* logarithms. (See [The Natural Logarithm].) In mathematical *parlance*, the common logarithm of x is denoted by $\log x$ while the natural logarithm is denoted by $\ln x$. If yet a different base b is used, it is denoted by $\log_b x$ as used earlier.

As an illustration of exchanging bases, again look at $y = 2^{3.659}$. Taking the logarithm using Base 2,

$$\log_2 y = 3.659$$

and since $\log_2 y = \log y / \log 2$, then

$$\log y = \log_2 y \times \log 2 = 3.659 \times \log 2 = 3.659 \times 0.30103 = 1.10147 = 0.10147 + 1.$$

The number with common logarithm 0.10147 is 1.2632, and so it must be that $y = 12.632$ as we found previously. (See Exponents and Powers.)

As another example, suppose we also needed the *natural* logarithm of our result $y = 12.632$. We can do so by noting that

$$\ln y = \log y / \log e.$$

For reference, the common logarithm of e is $\log e = 0.43429$, and, when needed, the natural logarithm of 10 is $\ln 10 = 2.30259$. Hence,

$$\ln y = 1.10147/0.434 = 2.536$$

and, correspondingly, $y = e^{2.536} = 12.632$.

Summary Thus Far

The basic rules of logarithms, true for any base b, can be summarized as follows:

- $\log_b 1 = 0$
- $\log_b b = 1$
- $\log_b(x \times y) = \log_b x + \log_b y$

Additionally, from these rules we found that:

- $\log_b x^r = r \times \log_b x$
- $\log_b x = \log_a x / \log_a b$
- $\log_b(1/x) = \log_b x^{-1} = -\log_b x$
- $\log_b(x/y) = \log_b x - \log_b y$

And, by standard convention:

- Notation when using Base 10: $\log x \equiv \log_{10} x$
- Notation when using Base e: $\ln x \equiv \log_e x$

So far we have discussed the nature of logarithms and their use in performing several types of calculations, assuming that the values of the necessary logarithms have been tabulated. However, though we have talked about a few specific examples, we have not addressed how the value of a logarithm for any arbitrary number can be obtained numerically. The history of calculating logarithms goes back over 400 years to the late-1500s, and the techniques of that day are long and laborious. During the following century, after the development of the *calculus* by Isaac Newton of England and Gottfried Wilhelm Leibniz of Germany, new approaches to such problems made the computation of a logarithm much more tractable. A *calculus*-based approach is presented in the following chapter from which formulas for directly computing logarithms will be presented.